Exercise set 11

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$\boxed{1}$ Let $|f|_{1,p} = (\int |\nabla f|^p)^{1/p}$. Prove that there is $C > 0$ such that for all $f \in W_0^{1,p}$,

$$|f|_{1,p} \leq \|f\|_{1,p} \leq C|f|_{1,p}.$$

What is C? Hint: Poincare.

$\boxed{2}$ (Evans P8 p 307) Show that there can be no trace operator for general functions in L^p, i.e. a bounded linear operator $T : L^p(\Omega) \to L^p(\partial\Omega)$ such that $Tf = f|_{\partial\Omega}$ for every $f \in L^p(\Omega) \cap C(\overline{\Omega})$.

Hint: Let $\Omega = B(0, 1)$, the unit ball, and define $f_n(r) = \min(n, -\log(1 - r))^{\frac{1}{p}}$ in polar coordinates. Show $0 \leq f \in C(\overline{\Omega})$ and that

$$\frac{\|f_n\|^p_{L^p(\partial\Omega)}}{\|f_n\|^p_{L^p(\Omega)}} \to \infty,$$

and hence T can not be bounded.

$\boxed{3}$ (Evans P4 p 306) Let $d = 1, 1 \leq p < \infty$, and $f \in W^{1,p}(0, 1)$. Show that

$$|f(x) - f(y)| \leq (\int_0^1 |f'|_p dx)^{\frac{1}{p}} |x - y|^{1-\frac{1}{p}}.$$

Hint: Show $f(x) = \int_0^x f'(x)dx + C =: g(x)$ *a.e.* for a constant C [They have the same weak/distributional derivatives...]. Check that g is absolutely continuous. Use the fundamental theorem of calculus and Hölder's inequality.

$\boxed{4}$ Interpolation inequalities (Evans P10 p 307). Let $\Omega \subset^d$ be open, bounded.

a) Integrate by parts to prove that for $2 \leq p < \infty$ and all $f \in C_c^\infty(\Omega)$,

$$\|\nabla f\|_{p,\Omega} \leq C\|f\|^{1/2}_{p,\Omega} \|\nabla^2 f\|^{1/2}_{p,\Omega}.$$

Hint: $\int_\Omega |\nabla f|^p = \sum_{i=1}^d \int_\Omega f_{x_i} f_{x_i} |\nabla f|^{p-2}$.

b) Assume also $p = 2$ and $\partial\Omega$ is C^2. Prove by approximation that the interpolation inequality of part (a) holds also for $f \in W^{2,2} \cap W_0^{1,2}(\Omega)$
Hint: You may use that there exist approximations $f_m \in C^2(\overline{\Omega})$ converging to f in $W^{2,2}$. Use also a sequence $g_m \in C_c^\infty$ converging to f in $W^{1,2}$ [why can you find such a sequence?]. Redo proof the of (a).

c) Prove that for $1 \leq p < \infty$ and $f \in C_c^\infty(\Omega)$,

$$\|\nabla f\|_{2p,\Omega} \leq C\|f\|_{\infty,\Omega}^{1/2}\|\nabla^2 f\|_{p,\Omega}^{1/2}.$$