Norwegian University of Science and Technology
Department of Mathematical Sciences

MA8105
Spring 2019

Solutions to exercise set 12

1 We first show that $C^{0,\gamma}$ is a normed vector space.

$$||f||_{0,\gamma} = ||f||_\infty + |f|_{0,\gamma}$$

is positive and if $f = 0$ then $||f||_{0,\gamma} = 0$. If $||f||_{0,\gamma} = 0$ then in particular $||f||_\infty = 0$ which implies $f = 0$.

$$||\alpha f||_{0,\gamma} = ||\alpha f||_\infty + |\alpha f|_{0,\gamma} = |\alpha|||f||_\infty + \sup_{x\neq y, x,y\in\bar{\Omega}} \frac{|\alpha f(x) - \alpha f(y)|}{|x-y|^\gamma} = |\alpha|||f||_{0,\gamma}.$$

Finally the triangle inequality holds:

$$\frac{|f(x) - f(y) + g(x) - g(y)|}{|x-y|^\gamma} \leq \frac{|f(x) - f(y)|}{|x-y|^\gamma} + \frac{|g(x) - g(y)|}{|x-y|^\gamma}$$

taking supremum over $x, y \in \bar{\Omega}$, $x \neq y$ shows that $|f+g|_{0,\gamma} \leq |f|_{0,\gamma} + |g|_{0,\gamma}$ and also $||f+g||_{0,\gamma} \leq ||f||_{0,\gamma} + ||g||_{0,\gamma}$.

$C^{0,\gamma}$ is a vector space: If $f, g \in C^{0,\gamma}$ and $\alpha, \beta \in \mathbb{R}$ then

$$\begin{aligned}|\alpha f(x) + \beta g(x) - (\alpha f(y) + \beta g(y))| &\leq |\alpha||f(x) - f(y)| + |\beta||g(x) - g(y)| \\ &\leq |\alpha|C_1|x-y|^\gamma + |\beta|C_2|x-y|^\gamma \leq C|x-y|^\gamma\end{aligned}$$

which shows that $\alpha f + \beta g \in C^{0,\gamma}$.

Finally we show that $C^{0,\gamma}$ is complete. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $C^{0,\gamma}(\bar{\Omega})$. Since

$$||f_n - f_m||_{0,\gamma} = ||f_n - f_m||_\infty + |f_n - f_m|_{0,\gamma}$$

we see that $\{f_n\}$ is Cauchy in $C(\bar{\Omega})$ so there exists a function $f \in C(\bar{\Omega})$ such that

$$||f_n - f||_\infty \to 0$$

as $n \to \infty$, since the space $C(\bar{\Omega})$ with $|| - ||_\infty$ is complete. We use this f as the candidate for the limit. We need to show that we have convergence to f in $C^{0,\gamma}(\bar{\Omega})$ and that $f \in C^{0,\gamma}(\bar{\Omega})$.

If $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$\sup_{x \neq y, x, y \in \bar{\Omega}} \frac{|f_n(x) - f_m(x) - (f_n(y) - f_m(y))|}{|x-y|^\gamma} < \epsilon$$

for all $m, n > N$. Hence, for any $x, y \in \bar{\Omega}$, $x \neq y$ we may pass to the limit $m \to \infty$ to obtain

$$\frac{|f_n(x) - f_n(y) - (f(x) - f(y))|}{|x-y|^\gamma} < \epsilon$$

Taking supremum over $x, y \in \bar{\Omega}$ gives $|f - f_n|_{0,\gamma} \to 0$ and hence $||f - f_n||_{0,\gamma} \to 0$.
To show that $f \in C^{0,\gamma}(\bar{\Omega})$ we use the triangle inequality $||a| - |b|| \leq |a - b|$,

$$\frac{|f(x) - f(y)|}{|x-y|^\gamma} \leq \epsilon + \frac{|f_n(x) - f_n(y)|}{|x-y|^\gamma}.$$

Since $f_n \in C^{0,\gamma}(\bar{\Omega})$, it follows that the limit function $f \in C^{0,\gamma}(\bar{\Omega})$.

2 **a)** We prove the following slightly more general interpolation inequality: Let $0 \leq \gamma_1 < \gamma < \gamma_2 \leq 1$ and $f \in \mathrm{C}^{0,\gamma_2}(\overline{\Omega})$. Then

$$\|f\|_{\mathrm{C}^{0,\gamma}(\overline{\Omega})} \leq \|f\|^{1-t}_{\mathrm{C}^{0,\gamma_1}(\overline{\Omega})} \|f\|^t_{\mathrm{C}^{0,\gamma_2}(\overline{\Omega})},$$

where $t \in (0,1)$ satisfies $\gamma = (1-t)\gamma_1 + t\gamma_2$. (In fact, $t = (\gamma - \gamma_1)/(\gamma_2 - \gamma_1)$.)
Note that the inequality in the problem text is obtained by taking $\gamma_1 = 0$ and noting that $|f|_{C^{0,0}(\overline{\Omega})} = \sup_{x,y \in \overline{\Omega}} |f(x) - f(y)| \leq 2 \sup_{x \in \overline{\Omega}} |f(x)| = 2\|f\|_{C_b(\overline{\Omega})}$.
Proof: Observe first that

$$\frac{|f(x) - f(y)|}{|x-y|^\gamma} = \left(\frac{|f(x) - f(y)|}{|x-y|^{\gamma_1}} \right)^{1-t} \left(\frac{|f(x) - f(y)|}{|x-y|^{\gamma_2}} \right)^t,$$

so

$$|f|_{\mathrm{C}^{0,\gamma}(\overline{\Omega})} \leq |f|^{1-t}_{\mathrm{C}^{0,\gamma_1}(\overline{\Omega})} |f|^t_{\mathrm{C}^{0,\gamma_2}(\overline{\Omega})}.$$

Therefore

$$\|f\|_{\mathrm{C}^{0,\gamma}(\overline{\Omega})} \leq \|f\|^{1-t}_\infty \|f\|^t_\infty + |f|^{1-t}_{\mathrm{C}^{0,\gamma_1}(\overline{\Omega})} |f|^t_{\mathrm{C}^{0,\gamma_2}(\overline{\Omega})} =: a_1^{1-t} b_1^t + a_2^{1-t} b_2^t.$$

Next put $A = a_1 + a_2$ and $A_i = a_i / A$ to see that

$$a_1^{1-t} b_1^t + a_2^{1-t} b_2^t = A^{1-t} \left[A_1 \left(\frac{b_1}{A_1} \right)^t + A_2 \left(\frac{b_2}{A_2} \right)^t \right].$$

Since the map $x \mapsto x^t$ is concave for $t \in (0,1)$, and $A_1 + A_2 = 1$, we further estimate

$$A_1 \left(\frac{b_1}{A_1} \right)^t + A_2 \left(\frac{b_2}{A_2} \right)^t \leq (b_1 + b_2)^t.$$

In total, this gives the desired inequality.

b) First, $\mathrm{C}^{0,\gamma_2}(\overline{\Omega}) \hookrightarrow \mathrm{C}^{0,\gamma_1}(\overline{\Omega})$ from $\|\cdot\|_{\mathrm{C}^{0,\gamma_1}(\overline{\Omega})} \leq \|\cdot\|_{\mathrm{C}^{0,\gamma_2}(\overline{\Omega})}$. As regards compactness, let $\{f_n\}_n \subset \mathrm{C}^{0,\gamma_2}(\overline{\Omega})$ be uniformly bounded, that is, $\|f_n\|_{\mathrm{C}^{0,\gamma_2}(\overline{\Omega})} \leq M < \infty$ for all n. Hence $\sup_n \|f_n\|_\infty < \infty$ by definition of functions in Hölder spaces, and $\{f_n\}_n$ is equibounded. Moreover, $\{f_n\}_n$ is equicontinuous, which follows directly from[1]

$$|f_n(x) - f_n(y)| \leq |f_n|_{\mathrm{C}^{0,\gamma_2}(\overline{\Omega})}|x-y|^{\gamma_2} \lesssim |x-y|^{\gamma_2}.$$

Arzelà–Ascoli's theorem and compactness of $\overline{\Omega}$ then yield that there exists a subsequence $\{f_{n_k}\}_k$ converging uniformly to some f in $\mathrm{C}(\overline{\Omega})$, or equivalently, in $\mathrm{C}^{0,0}(\overline{\Omega})$. Observe also that

$$|f(x) - f(y)| = \lim_{k\to\infty} |f_{n_k}(x) - f_{n_k}(y)| \lesssim |x-y|^{\gamma_2},$$

so $f \in \mathrm{C}^{0,\gamma_2}(\overline{\Omega})$. Interpolating

$$\|f_{n_k} - f\|_{\mathrm{C}^{0,\gamma_1}(\overline{\Omega})} \leq \|f_{n_k} - f\|_{\mathrm{C}^{0,0}(\overline{\Omega})}^{1-\frac{\gamma_1}{\gamma_2}} \|f_{n_k} - f\|_{\mathrm{C}^{0,\gamma_2}(\overline{\Omega})}^{\frac{\gamma_1}{\gamma_2}} \lesssim \|f_{n_k} - f\|_{\mathrm{C}^{0,0}(\overline{\Omega})}^{1-\frac{\gamma_1}{\gamma_2}}$$

then shows that $f_{n_k} \to f$ in $\mathrm{C}^{0,\gamma_1}(\overline{\Omega})$, establishing compactness of the embedding.
(If Ω is not bounded, you can only conclude local uniform convergence from Arzela-Ascoli, but the the arguement still works as before).

3 Linearity of Φ' follows directly from Y' being a vector space:

$$\begin{aligned}
\big(x, \Phi'(y_1' + cy_2')\big)_{X,X'} &= (\Phi x, y_1' + cy_2')_{Y,Y'} \\
&= (\Phi x, y_1')_{Y,Y'} + c(\Phi x, y_2')_{Y,Y'} \\
&= (x, \Phi' y_1')_{X,X'} + c(x, \Phi' y_2')_{X,X'},
\end{aligned}$$

so $\Phi'(y_1' + cy_2') = \Phi' y_1' + c\Phi' y_2'$. As regards injectivity, let $y' \in \Phi'$, i.e.

$$0 = (x, \Phi' y')_{X,X'} = (\Phi x, y')_{Y,Y'} \quad \text{for all} \quad x \in X.$$

Since $\Phi(X)$ is dense in Y and y' is continuous,

$$0 = (y, y')_{Y,Y'} \qquad \text{for all} \qquad y \in Y,$$

i.e. $y' \equiv 0$ and Φ' is injective. Finally, boundedness—and also continuity because Φ' is linear—is a consequence of

$$\begin{aligned}
\|\Phi' y'\|_{X'} = \sup_{\|x\|_X=1} \big|(x, \Phi' y')_{X,X'}\big| &= \sup_{\|x\|_X=1} \big|(\Phi x, y')_{Y,Y'}\big| \\
&\leq \|y'\|_{Y'} \sup_{\|x\|_X=1} \|\Phi x\|_Y = \|y'\|_{Y'}\|\Phi\|.
\end{aligned}$$

[1] By $A \lesssim B$ we mean that $A \leq cB$ for some unimportant constant $c > 0$. This simplifies estimates.

$\boxed{4}$ It is clear that $\mathrm{W}^{k,2}(\Omega) \hookrightarrow \mathrm{W}^{k-1,2}(\Omega)$ for all k and d.

Rellich–Kondrachov's result yields that $\mathrm{W}^{1,2}(\Omega)$ is compactly embedded in $\mathrm{L}^2(\Omega)$ for all d, where we have used that higher L^p norms control lower ones for bounded sets Ω, that is, $\|\cdot\|_{\mathrm{L}^p(\Omega)} \lesssim_\Omega \|\cdot\|_{\mathrm{L}^q(\Omega)}$ for $q > p$.

Next, let $\{f_n\}_n \subset \mathrm{W}^{2,2}(\Omega)$ be uniformly bounded. Then both $\{f_n\}$ and $\{f_n'\}$ lie in $\mathrm{W}^{1,2}(\Omega)$. By compactness, there exists a subsequence $f_{n_m} \to f \in \mathrm{L}^2(\Omega)$, and thereafter a subsubsequence of derivatives $f'_{n_{m_\ell}} \to g \in \mathrm{L}^2(\Omega)$. By continuity of weak differentiation, $f' = g$ in $\mathrm{L}^2(\Omega)$, so $f_{n_{m_\ell}} \to f$ in $\mathrm{W}^{1,2}(\Omega)$. Hence, $\mathrm{W}^{2,2}(\Omega)$ is compactly embedded in $\mathrm{W}^{1,2}(\Omega)$.

Now proceed by induction for general k.