Norwegian University of Science
and Technology
Department of Mathematical Sciences

MA8105
Nonlinear PDEs and Sobolev spaces
Spring 2019

Exercise set 12

1 Let $\Omega \subset \mathbb{R}^d$ be open, $k \in \mathbb{N}$, and $\gamma \in (0, 1]$. Prove that $C^{0,\gamma}(\overline{\Omega})$ is a Banach space.

2 Let $\Omega \subset \mathbb{R}^d$ is an open bounded set, and $0 < \gamma_1 < \gamma_2 < 1$.

a) Prove the following interpolation inequality for Hölder spaces:

$$\|f\|_{C^{0,\gamma_1}(\Omega)} \leq \|f\|_{C_b(\Omega)}^{1-\frac{\gamma_1}{\gamma_2}} \|f\|_{C^{0,\gamma_2}(\Omega)}^{\frac{\gamma_1}{\gamma_2}}$$

b) Prove that $C^{0,\gamma_2}(\overline{\Omega})$ is compactly embedded in $C^{0,\gamma_1}(\overline{\Omega})$.
Hint: Use Arzela-Ascoli's theorem and part (a).

3 Let X, Y be normed spaces, $\Phi : X \to Y$ be a continuous embedding (embedding=linear, bounded, $1-1$), and $\Phi(X) \subset Y$ dense. Prove that the adjoint $\Phi' : Y' \to X'$ defined by

$$(\Phi x, y')_{Y,Y'} = (x, \Phi' y')_{X,X'}, \quad y' \in Y',$$

is a continuous embedding.

4 (Holden Ex 1, p 119) Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. Show that $W^{k,2}(\Omega)$ is compactly embedded in $W^{k-1,2}(\Omega)$ for $k \in \mathbb{N}$.

Hint: Use Rellich-Kondrachov, do it for all values of d.