MA8105
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Norwegian University of Science and Technology
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Solutions to exercise set 13

1 The weak solution $u \in \mathrm{W}^{1,2}(\mathbb{R}^d)$ of $u - \Delta u = f \in \mathrm{L}^2(\mathbb{R}^d)$ by definition satisfies

$$\langle u, \varphi \rangle_{\mathrm{W}^{1,2}} := \int_{\mathbb{R}^d} (u\varphi + \nabla u \cdot \nabla \varphi)\,\mathrm{d}x = \int_{\mathbb{R}^d} f\varphi\,\mathrm{d}x \quad \text{for all} \quad \varphi \in \mathrm{W}^{1,2}(\mathbb{R}^d).$$

In particular, this holds with $\varphi = u$, so that

$$\|u\|^2_{\mathrm{W}^{1,2}} = \left|\int f u\,\mathrm{d}x\right| = \|u\|_{\mathrm{W}^{1,2}} \frac{|f(u)|}{\|u\|_{\mathrm{W}^{1,2}}} \le \|u\|_{\mathrm{W}^{1,2}} \sup_{0 \neq \phi \in \mathrm{W}^{1,2}} \frac{|f(\phi)|}{\|\phi\|_{\mathrm{W}^{1,2}}} = \|u\|_{\mathrm{W}^{1,2}} \|f\|_{(\mathrm{W}^{1,2})'},$$

where we have identified $f \in \mathrm{L}^2$ with the regular distribution $\phi \mapsto \int f\phi$ in $(\mathrm{W}^{1,2})'$. Thus

$$\|u\|_{\mathrm{W}^{1,2}} \le \|f\|_{(\mathrm{W}^{1,2})'}.$$

By linearity, $\mathrm{D}^h_{+,i} u$ solves $u - \Delta u = \mathrm{D}^h_{+,i} f$ weakly for all h and $i = 1, \dots, d$, and so

$$\|\mathrm{D}^h_{+} u\|_{\mathrm{W}^{1,2}} \le \|\mathrm{D}^h_{+} f\|_{(\mathrm{W}^{1,2})'}$$

as well. Estimating

$$\left|\int \mathrm{D}^h_{+,i} f\,\phi\,\mathrm{d}x\right| = \left|\int f\,\mathrm{D}^h_{-,i}\phi\,\mathrm{d}x\right| \le \|f\|_{\mathrm{L}^2} \|\mathrm{D}^h_{-,i}\phi\|_{\mathrm{L}^2} \lesssim \|f\|_{\mathrm{L}^2} \|\partial_i \phi\|_{\mathrm{L}^2} \le \|f\|_{\mathrm{L}^2} \|\phi\|_{\mathrm{W}^{1,2}}$$

with help of Theorem 47 (a) from the lectures, then shows that

$$\|\mathrm{D}^h_{+} u\|_{\mathrm{W}^{1,2}} \le \|\mathrm{D}^h_{+} f\|_{(\mathrm{W}^{1,2})'} \lesssim \|f\|_{\mathrm{L}^2}$$

for all $h > 0$. Hence, invoking Theorem 47 (b), $u \in \mathrm{W}^{2,2}$.