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**Solutions to exercise set 2**

**1** **Show that $(c_0)' = l^1$.**

Note, that $(c_0)'$ is equipped with the sup-norm.

"$\subseteq$":

Let $y \in (c_0)'$, given by $\langle y, x\rangle = \sum_{n=1}^{\infty} y_n x_n$ for all $x \in c_0$.

Assume, that $y \notin l^1$, so $\sum_{n=1}^{\infty} |y_n| = \infty$, in particular: $\forall \epsilon > 0\ \exists N > 0 : \sum_{n=1}^{N} |y_n| > \epsilon$.

Define $z_i \in c_0$ by setting $(z_i)_n = \begin{cases} sign(y_n) & n \le i \\ 0 & n > i \end{cases}$. Observe, that $\|z_i\|_\infty = 1$.

Then $\langle y, z_i\rangle = \sum_{n=1}^{i} |y_n|$. The sequence $(\langle y, z_i\rangle)_{i\in\mathbb{N}}$ is unbounded as seen before, so $y$ is no bounded linear functional. This is a contradiction!

"$\supseteq$":

Let $y \in l^1$. Set $C = \sum_{n=1}^{\infty} |y_n|$.

For any $x \in c_0$ we have, that $|\sum_{n=1}^{\infty} y_n x_n| \le \sum_{n=1}^{\infty} |y_n||x_n| \le \sum_{n=1}^{\infty} |y_n|\|x\|_\infty = C\|x\|_\infty$. So $y$ defines a bounded linear functional on $c_0$.

**ERJ Remark:** It remains to show that the norms on $(c_0)'$ and $l^1$ are the same. A reference from the internet can found here:

`http://math.stackexchange.com/questions/678911/the-dual-space-of-c-is-ell1`.

**2** **Show that $l^\infty$ is not seperable:**

Let $D = \{x_i\}_{i\in\mathbb{N}} \subseteq l^\infty$ be any countable subset of $l^\infty$.

Define $y$ by setting $y_n = \begin{cases} 1 & (x_n)_n < 0 \\ -1 & (x_n)_n > 0 \end{cases}$.

Obviously $y \in l^\infty$ and $|y_n - (x_n)_n| > 1$.

Therefore $\|y - x_i\|_\infty = \sup_{n\in\mathbb{N}} |y_n - (x_i)_n| > |y_n - (x_n)_n| > 1\ \forall i \in \mathbb{N}$.

Hence $D$ can not be dense in $l^\infty$.

**3** **Show that $l^p$ is seperable for $p \in [1, \infty)$:**

Define $D = \{x \in l^p | x_i \in \mathbb{Q}, x_i \neq 0 \text{ only for finitely many } i\}$.

$D$ is countable:

Define $D_n = \{x \in l^p | x_i \in \mathbb{Q}, x_i = 0 \text{ for } i > n\}$. Observe, that $D_n \simeq \mathbb{Q}^n$. Hence $D_n$ is countable. Consequently $D = \bigcup_{n\in\mathbb{N}} D_n$ is countable as well.

$D$ is dense:

Let $y \in l^p$ and $\epsilon > 0$.

Since $\sum_{n=1}^{\infty} |y_n|^p < \infty$, there exists an $N$, such that $\sum_{n>N} |y_n|^p < \frac{\epsilon^p}{2}$.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find an $x \in D$ such that

- $|x_n - y_n|^p < \frac{\epsilon^p}{2N}$ for $n \leq N$
- $x_n = 0$ for $n > N$

We conclude, that $\|x - y\|_p^p = \sum_{n \leq N} |x_n - y_n|^p + \sum_{n>N} |x_i|^p \leq N \frac{\epsilon^p}{2N} + \frac{\epsilon^p}{2} = \epsilon^p$.

Taking the $p^{th}$ root on both sides leads to the result.

$\boxed{4}$ Let $(x_n)_{n \in \mathbb{N}}$, defined by $(x_n)_k = \delta_{n,k}$.

**a) Show, that $x_n$ does not converge weakly in $l^1$.**

Let $x' \in (l^1)'$ defined by $\langle x', y \rangle = \sum_{k=1}^{\infty} (-1)^k y_k$.
$|\langle x', y \rangle| \leq \sum_{k=1}^{\infty} |(-1)^k y_k| = \sum_{k=1}^{\infty} |y_k| = \|y\|_1$, so $x'$ is indeed bounded.
Then $\langle x', x_n \rangle = (-1)^n$ is not a convergent sequence, so $x_n$ can not converge weakly in $l^1$.

**b) Show, that $x_n$ is weak* convergent in $l^\infty$.**

First define weak* convergence:

A sequence $(x'_n \in X')_{n \in \mathbb{N}}$ converges in the weak* topology to $x' \in X'$ if $\langle x'_n, x \rangle \to \langle x', x \rangle \ \forall x \in X$. So weak* convergence is pointwise-convergence.

Note, that $(l^1)' = l^\infty$.

Show, that the weak* limit of $x_n$ is 0. For any $y = (y_k)_{k \in \mathbb{N}} \in l^1$ we have $\langle x_n, y \rangle = \sum_{k=1}^{\infty} \delta_{n,k} y_k = y_n$. As $y$ is summable, we have, that $y_k \to 0$.

Weak convergence is a type of convergence defined for sequences in a normed space. Weak* convergence is a type of convergence definded for sequences in a dual space.

Elements $x \in l^\infty$ can be seen as vectors or as functionals.

Weak convergence in $l^\infty$ is defined via elements of the dual $(l^\infty)'$, which is the *ba*-space (https://en.wikipedia.org/wiki/Ba_space). Weak* convergence in $l^\infty$ is definde via elements of $l^1$ as $(l^1)' = l^\infty$. Since the *ba* space is strictly larger than $l^1$, weak* convergence does not imply weak convergence in $l^\infty$.

$\boxed{5}$ **Prove that the weak limit is unique.**

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $X$, such that $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$ for $x, y \in X$.

So $\langle x', x_n \rangle \to \langle x', x \rangle$ and $\langle x', x_n \rangle \to \langle x', y \rangle$ in $\mathbb{R}$.

As limits are unique in $\mathbb{R}$, we have $\langle x', x \rangle = \langle x', y \rangle \ \forall x' \in X' \Rightarrow \langle x', x - y \rangle = 0 \ \forall x' \in X'$ by linearity.

Define the bounded linear functional $f : span\{x - y\} \to \mathbb{R}$ by $f(\lambda(x - y)) = \lambda \|x - y\|_X$. Due to the Hahn-Banach theorem, there exists a $F \in X'$, such that $F(x - y) = f(x - y) = \|x - y\|$.

Now $0 = \langle F, x - y \rangle = \|x - y\|$, so $x = y$.

**6 Exercise 5 in Holden, page 33**

First note, that $x_n(s) \to 0$ for all $s \in [0,1]$, but $\|x_n\|_\infty = 1 \ \forall n$, so the $x_n$ converge pointwise, but not uniformely and hence strongly or in norm in $C[0,1]$

Assume, that the sequence does not converge weakly, so there is a linear functional $f \in X'$ such that $\langle f, x_n \rangle \not\to 0$. Consequently, there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and a $\delta > 0$ such that $|\langle f, x_{n_k} \rangle| > \delta \ \forall k \in \mathbb{N}$.

We can assume, that $n_{k+1} > 2n_k$. Otherwise we omit the $n_k$, that do not fulfill this property.

Define $y_K = \sum_{k=1}^K x_{n_k}$. Now we prove that $\|y_K\|_\infty \leq 1 + \sum_{k=0}^K 2^{-k}$ by induction.

$\|y_1\|_\infty = \|x_{n_1}\|_\infty = 1$.

On the intervall $[0, \frac{2}{n_{K+1}}]$ we have:

$$|x_{n_{K+1}}(s)| \leq 1$$

$$|x_{n_K}(s)| \leq \frac{2}{n_{K+1}} n_K \leq \frac{2}{n_{K+1}} \frac{n_{K+1}}{2} = 1$$

$$\vdots$$

$$|x_{n_1}(s)| \leq \frac{2}{n_{K+1}} n_1 \leq \frac{2}{n_{K+1}} \frac{n_{K+1}}{2^K} = 2^{-K+1}$$

Adding this up, we can conclude $|y_{K+1}(s)| \leq 1 + \sum_{k=0}^{K+1} 2^{-k}$.

On the intervall $[\frac{2}{n_{K+1}}, 1]$ we have $x_{n_{K+1}}(s) = 0$. Therefore

$$|y_{K+1}(s)| = |y_K(s)| \leq 1 + \sum_{k=0}^{K} 2^{-k} \leq 1 + \sum_{k=0}^{K+1} 2^{-k}.$$

As the geometric series is converging, the $y_K$ are uniformely bounded: $\|y_k\|_\infty < 3$.

By linearity of $f$ we have $\langle f, y_K \rangle = \sum_{k=1}^K \langle f, x_{n_k} \rangle > K\delta$.

Therefore we can compute the operator norm of $f$:

$$\|f\| = \sup_{x \neq 0} \frac{|\langle f, x \rangle|}{\|x\|_\infty} \geq \sup_K \frac{|\langle f, y_K \rangle|}{\|y_K\|_\infty} \geq \sup_K \frac{K\delta}{3} = \infty$$

Consequently $f \notin X'$. This is a contradiction!