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Solutions to exercise set 3

$\boxed{3}$ The predual of ℓ^1 is c_0, the space of bounded sequences tending to 0 at infinity. Then $x_n \overset{*}{\rightharpoonup} 0$ in ℓ^1 since for any $y \in c_0$,

$$\langle x, y\rangle = \sum_k x_{n,k} y_k = y_n \to 0.$$

$\boxed{4}$ Let $x : [0, T] \to \mathbb{R}^n$ be the solution of the ODE $\dot{x} = f(x),\ x(0) = x_0$. We will work with the integral form

$$x(t) = x_0 + \int_0^t f(x(s))ds, \quad t \in [0, T]. \tag{1}$$

The corresponding forward Euler discretisation is

$$y(t) = y(n\Delta t) + (t - n\Delta t) f(y(n\Delta t)),\ t \in [n\Delta t, (n+1)\Delta t],$$

$\Delta t = \frac{T}{N}$, and $y(0) = x_0$. Note that y is a continuous function coinciding with the Euler approximation at the points $n\Delta t$. Assume that f is Lipschitz,

$$|f(x) - f(y)| \le L_f |x - y|, \quad x, y \in \mathbb{R}^n.$$

a) Problem: Show by a direct argument that

$$|y(t)| \le |x_0|\, e^{L_f T} + |f(0)| \int_0^T e^{L_f t}\, dt =: M.$$

Solution: Set $t_n = n\Delta t$.

$$|f(y(t_j))| \le |f(0)| + |f(y(t_j)) - f(0)| \le |f(0)| + L_f\, |y(t_j)|\,.$$

Using this recursively, we get

$$\begin{aligned}
|y(t_n)| &\le |y(t_{n-1})| + \Delta t\, |f(y(t_{n-1}))| \\
&\le |y(t_{n-1})| + \Delta t\, (L_f\, |y(t_{n-1})| + |f(0)|) \\
&= (1 + L_f \Delta t)\, |y(t_{n-1})| + \Delta t\, |f(0)| \\
&\le (1 + L_f \Delta t)\, (|y(t_{n-2})| + \Delta t\, (L_f\, |y(t_{n-2})| + |f(0)|)) + \Delta t\, |f(0)| \\
&\ \vdots \\
&\le (1 + L_f \Delta t)^n\, |x_0| + \sum_{m=0}^{n-1} (1 + L_f \Delta t)^m \Delta t\, |f(0)| \\
&\le |x_0|\, e^{L_f n \Delta t} + |f(0)|\, \Delta t \sum_{m=0}^{n-1} e^{L_f m \Delta t}.
\end{aligned}$$

Seeking a bound for all times $t \in [0, T]$, we may use that

$$\begin{aligned} |y(t)| &\le |y(T)| = |y(t_N)| \\ &\le |x_0|\, e^{L_f T} + |f(0)| \int_0^T e^{L_f t}\, dt. \end{aligned}$$

b) Problem: Show by a direct argument that

$$|y(t) - y(s)| \le |t - s| \max_{|r| \le M} |f(r)| =: K, \quad t \in [0, T]\,.$$

Solution: We may assume without any loss of generality that $0 \le s < t \le T$. Since s is strictly smaller than t, we can then find a discretization such that

$$t_{m-1} < s \le t_m < \cdots < t_n \le t < t_{n+1}.$$

Then

$$\begin{aligned} y(t) - y(s) &= y(t) - y_n + (y_n - y_{n-1}) + \cdots + (y_{m+1} - y_m) + y_m - y(s) \\ &= (t - t_n) f(y(t_n)) + \Delta t \sum_{k=m}^{n-1} f(y(t_k)) + (t_m - s) f(y(t_{m-1}), \end{aligned}$$

and hence

$$\begin{aligned} |y(t) - y(s)| &\le (|t - t_n| + (m - n)\Delta t + |t_m - s|) \max_{\tau \in [s,t]} |f(y(\tau))| \\ &\le |t - s| \max_{|r| \le M} |f(r)|. \end{aligned}$$

c) Problem: Let $\Delta t = \Delta t_N = \frac{T}{N}, N = 1, 2, 3, \ldots$ and $y = y_{\Delta t_N} = y_N$. Use the Arzelà–Ascoli theorem to find a subsequence of $\{y_N\}_N$ converging uniformly on $[0, T]$.

Solution: By a), there is an $M \ge 0$ such that

$$\|y_N\|_\infty \le M \quad \text{for all} \quad N,$$

and hence $\{y_N\}_N$ is equibounded. By b), there is a $K \ge 0$ such that

$$|y_N(t) - y_N(s)| \le K|t - s| \quad \text{for all} \quad N,$$

and hence the sequence is also equicontinuous. Hence, by the Arzelà–Ascoli theorem, there exists a subsequence $\{y_{N_k}\}_{N_k}$ and a continuous limit $\bar{y}$ such that

$$\lim_{N_k \to \infty} \|y_{N_k} - \bar{y}\|_\infty = 0,$$

where local uniform convergence implies uniform convergence since $[0, T]$ is compact.

d) Problem: Verify that the uniform limit $\tilde{y}$ of *any* subsequence of the sequence $\{y_N\}_N$ from (c) is a solution of (1). (Hence also the subsequence found in (c)).
Solution: Let $\{y_{N_k}\}_{N_k} \subset \{y_N\}_N$ and $\tilde{y}$ such that $\lim_{N_k \to \infty} \|y_{N_k} - \tilde{y}\|_\infty = 0$. Then

$$y_{N_k}(t) = x_0 + \Delta t_{N_k} \sum_{j=0}^{n-1} f(y_{N_k}(j\Delta t_{N_k})) + (t - n\Delta t_{N_k}) f(y_{N_k}(n\Delta t_{N_k})), \tag{2}$$

for $n\Delta t_{N_k} \leq t < (n+1)\Delta t_{N_k}$.
Since $(t - n\Delta t_{N_k}) < \Delta t_{N_k} \xrightarrow[N_k\to\infty]{} 0$,

$$\lim_{N_k\to\infty}(t - n\Delta t_{N_k})f(y_{N_k}(n\Delta t_{N_k})) = 0.$$

We now show that the term with the sum in (2) converge to $\int_0^t f(\tilde{y}(s))ds$. Let $t_j := j\Delta t_{N_k}$, then

$$\begin{aligned}
&\left|\Delta t_{N_k}\sum_{j=0}^{n-1} f(y_{N_k}(t_j)) - \int_0^t f(\tilde{y}(s))\,ds\right| \\
&= \left|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} f(y_{N_k}(t_j))\,ds - \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} f(\tilde{y}(s))\,ds\right| \\
&\leq \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} |f(y_{N_k}(t_j)) - f(\tilde{y}(s))|\,ds \\
&\leq L_f\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} |y_{N_k}(t_j) - \tilde{y}(s)|\,ds \\
&= L_f\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} |y_{N_k}(t_j) - y_{N_k}(s) + y_{N_k}(s) - \tilde{y}(s)|\,ds \\
&\leq L_f\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} (K|t_j - s| + \|y_{N_k} - \tilde{y}\|_\infty)\,ds \\
&\leq L_f\sum_{j=0}^{n-1}\left(K(\Delta t_{N_k})^2 + \Delta t_{N_k}\|y_{N_k} - \tilde{y}\|_\infty\right) \\
&= L_f n\left(K(\Delta t_{N_k})^2 + \Delta t_{N_k}\|y_{N_k} - \tilde{y}\|_\infty\right) \\
&\leq L_f T\left(K(\Delta t_{N_k}) + \|y_{N_k} - \tilde{y}\|_\infty\right) \xrightarrow[N_k\to\infty]{} 0.
\end{aligned}$$

Since x_0 is fixed and $\lim_{N_k\to\infty} y_{N_k}(t) = \tilde{y}(t)$ for every $t \in [0,T]$ by uniform convergence, we may pass to the limit in (2) to see that $\lim_{N_k\to\infty} y_{N_k}(t) = \tilde{y}(t)$ satisfies (1).

e) **Problem:** Since the ODE has a unique solution (f is Lipschitz), conclude that the whole sequence converges.
Solution: By (c) and (d), $x := \bar{y}$ (where $\bar{y}$ defined in (c)!) is a solution of (1). By uniqueness it is the only one.
Assume that the whole sequence $\{y_N\}_N$ does not converge to x. Then by definition of convergence there exists an $\varepsilon_0 > 0$ and a subsequence $\left\{y_{N_j}\right\}_{N_j}$ such that

$$\left|y_{N_j}(t) - x(t)\right| \geq \varepsilon_0 \quad \text{for all } N_j. \tag{3}$$

But since this subsequence is equibounded and equicontinuous by (a) and (b), we then have by the Arzelà–Ascoli theorem that there exists a subsubsequence $\left\{y_{N_{j_i}}\right\}_{N_{j_i}}$ which converges uniformly to a limit $\tilde{y}$. By (d), $\tilde{y} = x$, and this contradicts (3). Hence we conclude that the whole sequence converges to x.