Exercise set 3

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1 State and prove Proposition 2.12 p. 14 in the Holden note.

2 State and prove Proposition 2.14 p. 14 in the Holden note.

3 Let $\{x_n\}_n \subset \ell^1$ be defined by $x_{n,k} = 1$ when $n = k$ and 0 otherwise. Prove that $x_n \overset{*}{\rightharpoonup} 0$ in ℓ^1.

Note that by last weeks problems, it does not converge weakly in ℓ^1! *Hence we have an example showing that weak-* convergence is weaker than weak convergence.*

4 Let $x : [0, T] \to \mathbb{R}^n$ solve the ODE $\dot{x} = f(x)$, $x(0) = x_0$, or in integral form,

$$x(t) = x_0 + \int_0^t f(x(s))ds, \quad t \in [0, T].$$

The corresponding forward Euler discretisation is

$$y(t) = y(n\Delta t) + (t - n\Delta t)f(y(n\Delta t)), \; t \in [n\Delta t, (n + 1)\Delta t],$$

$\Delta t = \frac{T}{N}$, and $y(0) = x_0$. Note that y is a continuous function coinciding with the Euler approximation at the points $n\Delta t$. Assume that f is Lipschitz,

$$|f(x) - f(y)| \leq L_f|x - y|, \quad x, y \in \mathbb{R}^n.$$

Prove the convergence of this method through the following steps:

a) Show by a direct argument that

$$|y(t)| \leq |x_0|e^{L_f T} + |f(0)| \int_0^T e^{L_f t} dt =: M, \; t \in [0, T].$$

b) Show by a direct argument that

$$|y(t) - y(s)| \leq |t - s| \max_{|r| \leq M} |f(s)|, \; s, t \in [0, T].$$

c) Let $\Delta t = \frac{T}{N}, N = 1, 2, 3, \ldots$ and $y = y_{\Delta t} = y_N$.

Use the Arzela-Ascoli theorem to find a subsequence of $\{y_{N_k}\}_{N_k} \subset \{y_N\}_N$ and continuous function $\bar{y}$ such that $y_{N_k} \to \bar{y}$ uniformly on $[0, T]$.

d) Verify that the uniform limit $\tilde{y}$ of *any* subsequence $\{y_N\}_N$ from the Euler method is a solution of the ODE in integral form. (I.e. also the subsequence in c)).

e) Since the ODE has a unique solution (f is Lipschitz), conclude that the whole sequence converges.

Hint: Use the argument for the corollary/2nd part of the Eberlein-Smuljan theorem.