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Solutions to exercise set 4

1 The reflection operator for functions is defined as

$$\sigma \,:\, f_\sigma(x) = f(-x)$$

Let $T \in \mathcal{D}'$ be an arbitrary distribution defined by $f \in L^1_{loc}$. Then, $(T_f)_\sigma = T_{f_\sigma}$. If $\phi \in C_c^\infty$ is a test function, then

$$\begin{aligned}(T_f)_\sigma(\phi) &= T_{f_\sigma}\phi(x)\,dx\\ &= \int_{\mathbb{R}} f_\sigma(x)\phi(x)\,dx\\ &= \int_{\mathbb{R}} f(-x)\phi(x)\,dx\\ &= \int_{\mathbb{R}} f(x)\phi(-x)\,dx\\ &= T_f(\phi_\sigma)\end{aligned}$$

which follows from changing variables. We have motivated the following definition:

$$T_\sigma(\phi) = T(\phi_\sigma) \tag{1}$$

From the definition of even and odd functions, we get the following characterizations:

$$T \text{ is even} \quad \Longrightarrow \quad T_\sigma = T \tag{2a}$$
$$T \text{ is odd} \quad \Longrightarrow \quad T_\sigma = -T \tag{2b}$$

2 Let $T : C_c^\infty \to \mathbb{R}$ be linear. First, we assume that T is continuous on C_c^∞. Since $0 \in C_c^\infty$, T will also be continuous at 0, so there is nothing more to prove here.

Then, assume that $T \in \mathcal{D}'$ is continuous at 0. Let $\phi_n \longrightarrow \phi$ in C_c^∞ for $\phi \neq 0$. Since T is linear, and continuous at 0, we get

$$\lim_{n\to\infty}[T(\phi_n) - T(\phi)] = \lim_{n\to\infty} T(\phi_n - \phi) = T(0) = T(\phi_0) = 0$$

Hence, $T \in \mathcal{D}'$ is continuous for every $\phi \in C_c^\infty$. We have now proven that the following equivalence is true:

$$T \in \mathcal{D}' \text{ is continuous at } C_c^\infty \iff T \in \mathcal{D}' \text{ is continuous at } 0$$

$\boxed{3}$ Let $T_1, T_2 \in \mathcal{D}'$ and $\alpha_1, \alpha_2 \in \mathbb{C}$, and define a linear combination $\widetilde{T} = \alpha_1 T_1 + \alpha_2 T_2$ by

$$(\alpha_1 T_1 + \alpha_2 T_2)(\phi) := \alpha_1 T_1(\phi) + \alpha_2 T_2(\phi).$$

Then $\tilde{T}$ is linear:

$$\begin{aligned}
&\langle \alpha_1 T_1 + \alpha_2 T_2, \beta_1 \phi_1 + \beta_2 \phi_2 \rangle \\
&= \alpha_1 \langle T_1, \beta_1 \phi_1 + \beta_2 \phi_2 \rangle + \alpha_2 \langle T_2, \beta_1 \phi_1 + \beta_2 \phi_2 \rangle \\
&= \alpha_1 \beta_1 \langle T_1, \phi_1 \rangle + \alpha_1 \beta_2 \langle T_1, \phi_2 \rangle + \alpha_2 \beta_1 \langle T_2, \phi_1 \rangle + \alpha_2 \beta_2 \langle T_2, \phi_2 \rangle \\
&= \beta_1 \langle \alpha_1 T_1 + \alpha_2 T_2, \phi_1 \rangle + \beta_2 \langle \alpha_1 T_1 + \alpha_2 T_2, \phi_2 \rangle
\end{aligned}$$

$\tilde{T}$ is continuous: Assume that $\phi_n \longrightarrow \phi$ in C_c^∞:

$$\begin{aligned}
&\lim_{n\to\infty} [(\alpha_1 T_1 + \alpha_2 T_2)(\phi_n) - (\alpha_1 T_1 + \alpha_2 T_2)(\phi)] \\
&= \lim_{n\to\infty} (\alpha_1 T_1 + \alpha_2 T_2)(\phi_n - \phi) \\
&= 0
\end{aligned}$$

We have shown that $\widetilde{T} : C_c^\infty \longmapsto \mathbb{R}$ is a linear and continuous functional and hence an element of $\mathcal{D}'$. Thus, $\mathcal{D}'$ is closed under linear combinations and therefore is a vector space.

$\boxed{4}$ Let $\{h_j\}_{j=1}^\infty$ be a sequence of *Friedriech mollifiers* in $\mathbb{R}^n$, defined by $h_j(x) = j^n h(jx)$. Here, h is an arbitrary function with the following properties:

$$h : C_c^\infty \longmapsto [0,1] \quad , \quad \|h\|_{L^1(\mathbb{R}^n)} = 1 \quad , \quad \mathrm{supp}(h) \subset B(0,1) \tag{3}$$

where $B(0,1)$ is an open ball of radius 1 centered at the origin. We see that

$$\|h_j\|_{L^1(\mathbb{R}^n)} = 1 \quad , \quad \mathrm{supp}(h_j) \subset B(0,1/j) \tag{4}$$

Our main quantity of interest is the function $f \in L^1_{loc}(\mathbb{R}^n)$. We define its *regularizer* as the convolution

$$f_j(x) = (h_j * f)(x) = \int_{\mathbb{R}^n} h_j(y) f(x-y)\, dy \tag{5}$$

Let $R > 0$. By *Fubini's theorem* and the properties of h_j (4), we obtain

$$\begin{aligned}
&\int_{B(0,R)} |f(x) - f_j(x)|\, dx \\
&= \int_{B(0,R)} \left| f(x) \int_{|y|<1/j} h_j(y)\, dy - \int_{|y|<1/j} h_j(y) f(x-y)\, dy \right| dx \\
&= \int_{B(0,R)} \left| \int_{|y|<1/j} h_j(y) [f(x) - f(x-y)]\, dy \right| dx \\
&\le \int_{|y|<1/j} h_j(y) \int_{B(0,R)} |f(x) - f(x-y)|\, dx\, dy \\
&\le \|h_j\|_{L^1} \left(\sup_{|y|<1/j} \int_{B(0,R)} |f(x) - f(x-y)|\, dx \right)
\end{aligned}$$

By letting $j \longrightarrow \infty$ and using the continuity of translation in L^1:

$$\sup_{|y|<\epsilon} \int_{B(0,R)} |f(x) - f(x-y)|\, dx \to 0 \quad \text{as} \quad \epsilon \to 0$$

(holds for (uniformly) functions, and then by approximation for any L^1-function), we get

$$\lim_{j\to\infty} \int_{B(0,R)} |f(x) - f_j(x)|\, dx = 0. \tag{6}$$

The original given statement is

$$\int_\Omega f(x)\phi(x)\, dx = 0 \quad , \quad \forall \phi \in C_c^\infty \tag{7}$$

If $\phi(x) \ = h_j(y-x) \in C_c^\infty$, then for all $j \in \mathbb{N}$ and $y \in \mathbb{R}^n$, we obtain

$$f_j(y) = \int_\Omega f(x)h(y-x)\, dx = 0 \tag{8}$$

Combining the two identities (6) and (8) we get

$$\begin{aligned}
&\int_{B(0,R)} |f(x)|\, dx \\
&= \int_{B(0,R)} |f(x) - f_j(x) + f_j(x)|\, dx \\
&\leq \underbrace{\int_{B(0,R)} |f(x) - f_j(x)|\, dx}_{\longrightarrow 0} + \underbrace{\int_{B(0,R)} |f_j(x)|\, dx}_{=0} \\
&= 0
\end{aligned}$$

Thus, we have proven that for all $f \in L^1_{loc}$ and $\phi \in C_c^\infty$, the statement below is true:

$$\int_\Omega f(x)\phi(x)\, dx = 0 \implies f = 0 \text{ a.e. in } \mathbb{R}^n$$

$\boxed{5}$ T_3 **is well-defined:** Let $\phi \in C_c^\infty(0,2)$ be a test function, and $\text{supp}(\phi) \subset (0,1)$. Since $(0,1)$ is an open set, it implies that

$$\text{dist}(\text{supp}(\phi), 0) > 0$$

Therefore, it exists an $m \in \mathbb{N}$ depending on $\text{supp}(\phi)$ such that

$$n > m \quad \implies \quad \phi\left(\frac{1}{n}\right) = 0$$

In this way, T_3 becomes finite for all $\phi \in C_c^\infty$:

$$T_3(\phi) = \sum_{n=1}^{\infty} \int_0^1 \delta\left(x - \frac{1}{n}\right) \phi(x)\, dx = \sum_{n=1}^{m} \phi\left(\frac{1}{n}\right)$$

T_3 **is linear:** Let $\phi_1, \phi_2 \in C_c^\infty$, and $m \in \mathbb{N}$ such that

$$n > m \quad \Longrightarrow \quad \phi_1\left(\frac{1}{n}\right) = 0 \quad , \quad \phi_2\left(\frac{1}{n}\right) = 0$$

Then, T_3 is linear:

$$\begin{aligned}
T_3(\alpha_1\phi_1 + \alpha_2\phi_2) &= \sum_{n=1}^{\infty}\left[\alpha_1\phi_1\left(\frac{1}{n}\right) + \alpha_2\phi_2\left(\frac{1}{n}\right)\right] \\
&= \sum_{n=1}^{m}\left[\alpha_1\phi_1\left(\frac{1}{n}\right) + \alpha_2\phi_2\left(\frac{1}{n}\right)\right] \\
&= \alpha_1\sum_{n=1}^{m}\phi_1\left(\frac{1}{n}\right) + \alpha_2\sum_{n=1}^{m}\phi_2\left(\frac{1}{n}\right) \\
&= \alpha_1 T_3(\phi_1) + \alpha_2 T_3(\phi_2)
\end{aligned}$$

T_3 **is continuous:** Assume that $\phi_j \longrightarrow \phi$ in C_c^∞, $K \subset\subset (0,2)$ such that $\operatorname{supp}(\phi_j) \subset K$, and $m \in \mathbb{N}$ such that $\frac{1}{n} \notin K$ for $n > m$. Then

$$\lim_{j\to\infty} T_3(\phi_j - \phi) = \lim_{j\to\infty}\sum_{n=1}^{m}\left[\phi_j\left(\frac{1}{n}\right) - \phi\left(\frac{1}{n}\right)\right] = 0$$

Thus, we have proven that T_3 is a distribution on $\mathcal{D}'(0,2)$.

6 Let $T \in \mathcal{D}'$ be an arbitrary distribution, and $\boldsymbol{\alpha}$ is a multi-index such that $|\boldsymbol{\alpha}| < \infty$, and $\partial^{\boldsymbol{\alpha}} T$ is defined in the usual way.

$\partial^{\boldsymbol{\alpha}} T$ **is linear:** Since $\phi \in C_c^\infty \implies \partial^{\boldsymbol{\alpha}}\phi \in C_c^\infty$, for $\phi_1, \phi_2 \in C_c^\infty$ and linearity of T,

$$\begin{aligned}
&\langle \partial^{\boldsymbol{\alpha}} T, \alpha_1\phi_1 + \alpha_2\phi_2\rangle \\
&= (-1)^{|\boldsymbol{\alpha}|}\langle T, \alpha_1\partial^{\boldsymbol{\alpha}}\phi_1 + \alpha_2\partial^{\boldsymbol{\alpha}}\phi_2\rangle \\
&= \alpha_1(-1)^{|\boldsymbol{\alpha}|}\langle T, \partial^{\boldsymbol{\alpha}}\phi_1\rangle + \alpha_2(-1)^{|\boldsymbol{\alpha}|}\langle T, \partial^{\boldsymbol{\alpha}}\phi_2\rangle \\
&= \alpha_1\langle \partial^{\boldsymbol{\alpha}} T, \phi_1\rangle + \alpha_2\langle \partial^{\boldsymbol{\alpha}} T, \phi_2\rangle
\end{aligned}$$

$\partial^{\boldsymbol{\alpha}} T$ **is continuous:** Let $\phi_j \longrightarrow \phi$ in C_c^∞. By the continuity of T, we get

$$\begin{aligned}
&\langle \partial^{\boldsymbol{\alpha}} T, \phi_n\rangle \\
&= (-1)^{|\boldsymbol{\alpha}|}\langle T, \partial^{\boldsymbol{\alpha}}\phi_n\rangle \\
&\to (-1)^{|\boldsymbol{\alpha}|}\langle T, \partial^{\boldsymbol{\alpha}}\phi\rangle \\
&= \langle \partial^{\boldsymbol{\alpha}} T, \phi\rangle
\end{aligned}$$

We have finally proven the following statement:

$$T \in \mathcal{D}' \implies \partial^{\boldsymbol{\alpha}} T \in \mathcal{D}'$$

7 T **is well-defined:** Let $K \subset\subset \mathbb{R}$, and $\phi \in C_c^\infty$ such that $\operatorname{supp}(\phi) \subset K$. By the *Heine-Borel theorem*, a set in $\mathbb{R}$ is compact iff it is both closed and bounded. Thus,

ϕ and all of its derivatives vanish outside the bounded set K. There is an $m \in \mathbb{N}$ depending on $\operatorname{supp}(\phi)$ such that $K \cap \mathbb{N} = \{1, 2, 3, \dots, m\}$. This implies that

$$T(\phi) = \sum_{n=1}^{\infty} \phi^{(n)}(n) = \sum_{n=1}^{m} \phi^{(n)}(n)$$

Since ϕ is smooth and the sum above is finite, T is well-defined.

T is linear: Assume that $\phi_1, \phi_2 \in C_c^\infty$, and $m \in \mathbb{N}$ such that $n \notin \operatorname{supp}(\phi_1) \cup \operatorname{supp}(\phi_2)$ for $n > m$. By the linearity of derivatives, T is also linear:

$$\begin{aligned}
T(\alpha_1\phi_1 + \alpha_2\phi_2) &= \sum_{n=1}^{\infty} (\alpha_1\phi_1 + \alpha_2\phi_2)^{(n)}(n) \\
&= \sum_{n=1}^{m} (\alpha_1\phi_1 + \alpha_2\phi_2)^{(n)}(n) \\
&= \alpha_1 \sum_{n=1}^{m} \phi_1^{(n)}(n) + \alpha_2 \sum_{n=1}^{m} \phi_2^{(n)}(n) \\
&= \alpha_1 T(\phi_1) + \alpha_2 T(\phi_2)
\end{aligned}$$

T is continuous: Assume that $\phi_j \longrightarrow \phi$ in C_c^∞, and $m \in \mathbb{N}$ such that

$$n > m \qquad \Longrightarrow \qquad n \notin K \supset \bigcup_{j \in \mathbb{N}} \operatorname{supp}(\phi_j)$$

Then, T is continuous:

$$\lim_{j\to\infty} T(\phi_j - \phi) = \lim_{j\to\infty} \left[\sum_{n=1}^{m} (\phi_j - \phi)^{(n)}(n) \right] = 0$$

Thus, we have finally proven that $T \in \mathcal{D}'$ is a distribution.

$\boxed{8}$ Let $f(x) = \ln|x|$, and T_f is the associated distribution. We let $\phi \in C_c^\infty$ and integrate by parts to find the derivative $(T_f)'$:

$$\begin{aligned}
\langle (T_f)', \phi \rangle &= -\int_{-\infty}^{\infty} \ln|x| \phi'(x)\, dx \\
&= -\lim_{\epsilon\to 0} \left[\int_{-\infty}^{-\epsilon} \ln|x|\phi'(x)\, dx + \int_{\epsilon}^{\infty} \ln|x|\phi'(x)\, dx \right] \\
&= \underbrace{-\lim_{\epsilon\to 0} [\phi(-\epsilon) - \phi(\epsilon)] \ln|\epsilon|}_{=0} + \lim_{\epsilon\to 0} \left[\int_{-\infty}^{-\epsilon} \frac{\phi(x)}{x}\, dx + \int_{\epsilon}^{\infty} \frac{\phi(x)}{x}\, dx \right] \\
&= \lim_{\epsilon\to 0} \int_{|x|>\epsilon} \frac{\phi(x)}{x}\, dx \\
&= \left\langle PV\left(\frac{1}{x}\right), \phi \right\rangle
\end{aligned}$$

where PV is an abbreviation of the *Cauchy Principal Value*. The first term vanished because of the *Mean Value Theorem*:

$$\phi(-\epsilon) - \phi(\epsilon) \leq 2\epsilon\|\phi'\|_\infty$$
$$\lim_{\epsilon\to 0} \epsilon \ln(\epsilon) = 0$$

Thus, the derivative of $\ln|x|$, in the sense of distributions, is

$$PV\left(\frac{1}{x}\right) \tag{9}$$