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Solutions to exercise set 5

$\boxed{1}$ For $S, T \in D'$, both compactly supported, we define $C_S T(\phi) = T(S_\sigma * \phi)$. Here, $S_\sigma(\phi) = S(\phi_\sigma)$ for $\phi \in C_c^\infty$. Then $C_S T \in D'$:

-*well defined:* We note that $S_\sigma * \phi \in C_c^\infty$ and T is well defined.

-*linearity:* Follows from the linearity (bilinearity) of the convolution and that of T.

-*continuity:* Suppose $\phi_n \to \phi$ in C_c^∞. Then $S_\sigma * \phi_n \to S_\sigma * \phi$ in C_c^∞. Hence, since T is continuous:

$$C_S T(\phi_n) = T(S_\sigma * \phi_n) \to T(S_\sigma * \phi) = C_S T(\phi).$$

Now we show that convolutions commute

$$C_S T * \psi = C_T S * \psi \quad \text{in } D'.$$

for any $\psi \in C_c^\infty$. First, we need an associative property of the convolution between functions and distributions.

$$T * (\phi * \psi) = (T * \phi) * \psi.$$

To see this, we use the continuity and linearity of T and approximate the convolution (in one dimension) of functions with a Riemann sum:

$$\begin{aligned}
&(T * (\phi * \psi))(x) = \tau_x T((\phi * \psi)_\sigma) = T\left((\phi * \psi)(x - .)\right) \\
&= T\left(\lim_{\Delta y \to 0} \sum_{i \le N\Delta y} \phi(x - i\Delta y - .)\psi(i\Delta y)\Delta y\right). \\
&= \lim_{\Delta y \to 0} \sum_{i \le N\Delta y} (T * \phi)(x - i\Delta y)\psi(i\Delta y)\Delta y \\
&= \int (T * \phi)(x - y)\psi(y)dy = \left((T * \phi) * \psi\right)(x).
\end{aligned}$$

We can now calculate

$$\begin{aligned}
C_\psi(C_S T(\phi)) &= C_S T(\phi * \psi_\sigma) = T(S_\sigma * (\phi * \psi_\sigma)) \\
&= T((S_\sigma * \phi) * \psi_\sigma) = C_\psi T(S_\sigma * \phi).
\end{aligned}$$

Now, we use the fact that $C_\psi T$ is a *regular distribution* with "density" $\tau_x T(\psi_\sigma)$,

$$\begin{aligned}
C_\psi(C_S T(\phi)) &= \int \tau_x T(\psi_\sigma)(S_\sigma * \phi)(x)dx \\
&= \int \tau_x \left(S_\sigma(\phi_\sigma)\right) \tau_x T_\sigma(\psi)dx \\
&= \int \tau_x S(\phi)(T_\sigma * \psi_\sigma)(x)dx = C_{\phi_\sigma} S(T_\sigma * \psi_\sigma) \\
&= S((T_\sigma * \psi_\sigma) * \phi) \\
&= S(T_\sigma * (\psi_\sigma * \phi)) \\
&= S(T_\sigma * (\phi * \psi_\sigma)) \\
&= S((T_\sigma * \phi) * \psi_\sigma) \\
&= C_\psi \left(C_T S(\phi)\right).
\end{aligned}$$

Here, we used the associative property of convolution between functions and distributions, and the commutative property of convolution with functions. Hence, for any $\psi \in C_c^\infty$,

$$C_S T * \psi = C_T S * \psi.$$

Now take $\psi \in C_c^\infty$ with $\int \psi = 1$ and put $\psi_n(x) = n^d \psi(nx)$. Then

$$\begin{aligned}
C_S T * \psi_n &\to C_S T \\
C_T S * \psi_n &\to C_T S
\end{aligned}$$

in D'. We see that

$$C_S T = \lim_{n\to\infty} C_S T * \psi_n = \lim_{n\to\infty} C_T S * \psi_n = C_T S.$$

$\boxed{2}$ Take $\phi \in C_c^\infty$. We may assume

$$\operatorname{supp} \phi \subset B(0,r).$$

Then

$$|T_f - T_{f_n}| \le \int_{|x|<r} |f(x) - f_n(x)||\phi(x)|dx \le ||\phi||_\infty \int_{|x|<r} |f(x) - f_n(x)|dx \to 0$$

as $n \to \infty$. This shows that $f_n \to f$ in D'.

$\boxed{3}$ We want to show that

$$\phi_n := \eta(\frac{x}{n})(\psi_n * T) \to T \quad \text{in } D'$$

when $\eta, \psi \in C_c^\infty$, knowing that the result is true if η is replaced by 1. Take $\phi \in C_c^\infty$. Again, we may assume

$$\operatorname{supp} \phi \subset B(0,r).$$

Take n large so that $\eta(\frac{x}{n}) = 1$ in $B(0,r)$. Then, for large n,

$$T_{\phi_n} = \int \eta(\frac{x}{n})(\psi_n * T)(x)\phi(x)dx = \int_{|x|<r} (\psi_n * T)(x)\phi(x)dx \to T$$

in D' since $(\psi_n * T) \to T$.

$\boxed{4}$ Using integration by parts,

$$\begin{aligned}\partial_i T(\phi) &= \lim_{n\to\infty} \int \partial_i \psi_n(x)\phi(x)dx \\ &= \lim_{n\to\infty} \left([\phi(x)\psi_n(x)]_*^{**} - \int \psi_n(x)\partial_i\phi(x)dx \right) \\ &= -\lim_{n\to\infty} \int \psi_n(x)\partial_i\phi(x)dx = -T(\partial_i\phi),\end{aligned}$$

the boundary term is zero since $\psi_n, \phi \in C_c^\infty$. The last equality follows by the definition of T. If $\psi_n, \eta_n \to T$ in D' with $\psi_n, \eta_n \in C_c^\infty$ then

$$\begin{aligned}\lim_{n\to\infty} \int \partial_i\psi_n(x)\phi(x)dx &= -\lim_{n\to\infty} \int \psi_n(x)\partial_i\phi(x)dx = -T(\partial_i\phi) \\ &\overset{*}{=} -\lim_{n\to\infty} \int \eta_n(x)\partial_i\phi(x)dx = \lim_{n\to\infty} \int \partial_i\eta_n(x)\phi(x)dx\end{aligned}$$

$*$: $\psi_n, \eta_n \to T$ in D'. The definition is therefore independent on the choice of the approximating sequence.

$\boxed{5}$ $u \in L^1_{\text{loc}}(\mathbb{R})$:

$$\int_{-\infty}^{\infty} \frac{1}{2} e^{-|x|} dx = 2\int_0^\infty \frac{1}{2} e^{-x} dx = 1 < \infty.$$

(In fact, $u \in L^1(\mathbb{R})$).

First note that

$$(1 - \partial^2)\frac{1}{2} e^{-|x|} = 0$$

when $x \neq 0$. Further, note that u is not a C^1 function on any interval containing $x = 0$:

$$\lim_{x\to 0^+} u'(x) = -\frac{1}{2}, \quad \lim_{x\to 0^-} u'(x) = \frac{1}{2} \tag{1}$$

Using integration by parts, we find for any $\epsilon > 0$,

$$\int_{|x|>\epsilon} u(x)\phi''(x)dx = u(-\epsilon)\phi'(-\epsilon) - u(\epsilon)\phi'(\epsilon) + u'(\epsilon)\phi(\epsilon) - u'(-\epsilon)\phi(-\epsilon) + \int_{|x|>\epsilon} u''(x)\phi(x)dx.$$

The boundary terms at infinity gives zero contribution since ϕ is compactly supported. Hence

$$\begin{aligned}\int_{|x|>\epsilon} u(x)\left(\phi(x)-\phi''(x)\right)dx = \int_{|x|>\epsilon} \phi(x)\left(u(x)-u''(x)\right)dx \\ + \left[u(\epsilon)\phi'(\epsilon)-u(-\epsilon)\phi'(-\epsilon)\right] \\ + \left[u'(-\epsilon)\phi(-\epsilon)-u'(\epsilon)\phi(\epsilon)\right]\end{aligned}$$

The first term is zero for any $\epsilon > 0$, since u solves the equation when $|x| > \epsilon$. The second term goes to zero as $\epsilon \to 0$ since u is *continuous* in $\mathbb{R}$. For the last term we use equation (1)

$$\lim_{\epsilon\to 0}\left[u'(-\epsilon)\phi(-\epsilon)-u'(\epsilon)\phi(\epsilon)\right] = \frac{\phi(0)}{2} - \frac{-\phi(0)}{2} = \phi(0).$$

In total,

$$\lim_{\epsilon\to 0}\int_{|x|>\epsilon} u(x)\left(\phi(x)-\phi''(x)\right)dx = \phi(0),$$

which shows that u is a fundamental solution of $L = 1-\partial^2$.

$\boxed{6}$ We solve

$$T'' - 2T' = \delta'' \quad \text{in } D'.$$

First, let $V = T' - 2T - \delta'$. We see that $V' = 0$ in D':

$$V'(\phi) = -V(\phi') = T'(\phi') - 2T(\phi') - \delta'(\phi') = T''(\phi) - 2T(\phi) - \delta''(\phi) = 0$$

for any $\phi \in C_c^\infty$. Hence, V is a constant in D' and

$$T' - 2T = \delta' + K.$$

Since the equation is *linear*, we may first solve $T' - 2T = K$, then solve $T' - 2T = \delta'$ and add the solutions. Using an integrating factor e^{-2t}, the solution of the first equation is $T = -\frac{K}{2} + K_2 e^{2t}$. For the second equation, let $U = e^{-2t}T$. Then

$$\begin{aligned}U'(\phi) &= -U(\phi') = -T(e^{-2t}\phi') = -T((e^{-2t}\phi)' + 2e^{-2t}\phi) \\ &= T'(e^{-2t}\phi) - 2T(e^{-2t}\phi) = \delta'(e^{-2t}\phi).\end{aligned}$$

Here, the definition of derivative, the equation and the linearity of T was used. Here,

$$\delta'(e^{-2t}\phi) = -\delta((e^{-2t}\phi)') = -\delta(e^{-2t}\phi' - 2e^{-2t}\phi) = -\phi'(0) + 2\phi(0) = \delta'(\phi) + 2\delta(\phi).$$

Let $W = \delta + 2H$, where H is the Heaviside step function. Then

$$(U - W)' = 0 \quad \text{in } D'.$$

So $U = W$ pluss a constant in D'. We can now invert $T = e^{2t}U$ to find the solution of the second equation. In total we have

$$T = (\delta + 2H)e^{2t} + K_1 + K_2 e^{2t} = \delta + 2He^{2t} + K_1 + K_2 e^{2t}.$$

In the last equality we used the fact that $(\delta e^{2t})(\phi) = \delta(\phi)$. In terms of test functions,

$$T(\phi) = \phi(0) + 2\int H(t)e^{2t}\phi(t)dt + K_1\int \phi(t)dt + K_2\int e^{2t}\phi(t)dt.$$

One can check that this solves $T''(\phi) - 2T'(\phi) = \phi''(0)$. Here, we only check that $T = \delta + 2He^{2t}$ solves $T' - 2T = \delta'$ in D':

$$\begin{aligned}
&T(\phi) = \phi(0) + 2\int H(t)e^{2t}\phi(t)dt,\\
&T'(\phi) = -T(\phi') = -\phi'(0) - 2\int_0^\infty e^{2t}\phi'(t)dt\\
&= \delta'(\phi) - 2\left[e^{2t}\phi(t)\right]_{t=0}^{\infty} + 4\int_0^\infty e^{2t}\phi(t)dt\\
&= \delta'(\phi) + 2\phi(0) + 4\int H(t)e^{2t}\phi(t)dt.
\end{aligned}$$