MA8105

Nonlinear PDEs and Sobolev spaces

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Exercise set 5

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Department of Mathematical Sciences

1 Prove that $C_S T = C_T S$ for all $S, T \in D'$ with compact support.

2 Prove that $f, f_n \in L^1_{loc}$ and $\int_{|x|<R} |f(x) - f_n(x)| dx \to 0$ for all $R > 0$, implies that $f_n \to f$ in D'.

3 Prove that $\eta(\frac{x}{n})(\psi_n * T) \to T$ in D' when $\eta, \psi \in C_c^\infty$, $\eta = 1$ for $|x| < 1$, $\psi \geq 0$, $\int \psi = 1$, $\psi_n(x) = n^d \psi(nx)$.

Hint: Use that fact that the result holds if η is replaced by 1.

4 Define D' as the D' limits of C_c^∞ functions ($T \in D'$ if there is $\{\psi_n\} \subset C_c^\infty$ such that $\psi_n \to T$ in D'). By a theorem in class, such a limit is a continuous and linear functional on C_c^∞. Define the derivative of T in the following way:

$$\partial_i T(\phi) = \lim_n \int \partial_i \psi_n \phi \, dx.$$

Show that then $\partial_i T(\phi) = -T(\partial_i \phi)$ by passing to the limit. Conclude that this definition of derivative does not depend on the approximating sequence $\{\psi_n\}_n$.

5 Show that $\frac{1}{2} e^{-|x|}$, $x \in \mathbb{R}$, belongs to L^1_{loc} and is a fundamental solution of $L = 1 - \partial^2$, i.e.

$$\int u(1 - \partial^2)\phi \, dx = \phi(0) \quad \text{for all} \quad \phi \in C_c^\infty.$$

Hint: Use similar ideas as in the Holden note: truncation of domain and integration by parts.

6 Solve the equation

$$T'' - 2T' = \delta'' \quad \text{in} \quad D'.$$

Hint: Integrate once, then use integrating factor. The answer should be $T = \delta + 2e^{2t} H + K_1 + e^{2t} K_2$ where K_1, K_2 are arbitrary constants.