Norwegian University of Science and Technology
Department of Mathematical Sciences

Solutions to exercise set 6

In the following, for a function $f \in L^p(X,\mu)$ we shall for simplicity use the notation $L^p(X) = L^p(X,\mu)$ and $\|f\|_p := \|f\|_{L^p(X)}$ for some space X and honest measure μ, while λ will denote the Lebesgue measure.

$\boxed{1}$ We are to prove the Hölder inequality

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

for $f \in L^p(X)$, $g \in L^q(X)$, $1/p + 1/q = 1$ and $p, q \in [1, \infty]$.

Let us first treat the case $p = \infty$, $q = 1$. Then it is straightforward to show

$$\|fg\|_1 = \int_X |fg| \, d\mu \leq \|f\|_\infty \int_X |g| \, d\mu = \|f\|_\infty \|g\|_1.$$

Now we treat $p, q \in (1, \infty)$ and for this setting we first recall Young's inequality,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \text{ for } a, b \geq 0 \text{ and } \frac{1}{p} + \frac{1}{q} = 1.$$

Now define $\tilde{f} := f/\|f\|_p$ and $\tilde{g} := g/\|g\|_q$. Then we have by Young's inequality

$$\|\tilde{f}\tilde{g}\|_1 = \int_X |\tilde{f}\tilde{g}| \, d\mu \leq \frac{1}{p} \int_X |\tilde{f}|^p \, d\mu + \frac{1}{q} \int_X |\tilde{g}|^q \, d\mu = \frac{1}{p} + \frac{1}{q} = 1,$$

which is equivalent to

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

$\boxed{2}$ We are to prove the Minkowski inequality

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

for $f, g \in L^p(X)$, $p \in [1, \infty]$. We first treat $p = \infty$, where we have

$$\|f + g\|_\infty = \operatorname*{ess\,sup}_{x \in X} |(f + g)(x)| \leq \operatorname*{ess\,sup}_{x \in X} (|f(x)| + |g(x)|) = \|f\|_\infty + \|g\|_\infty,$$

by the triangle inequality. Similarly for $p = 1$ we have

$$\|f + g\|_1 = \int_X |f + g| \, d\mu \leq \int_X (|f| + |g|) \, d\mu = \|f\|_1 + \|g\|_1.$$

Finally we treat $p \in (1,\infty)$ where we first note that $1/p + 1/q = 1$ implies $q = p/(p-1)$. Then yet another application of the triangle inequality followed by the Hölder inequality gives

$$\begin{aligned}
\|f+g\|_p^p &= \int_X |f+g|^p \, d\mu \le \int_X |f+g|^{p-1}|f| \, d\mu + \int_X |f+g|^{p-1}|g| \, d\mu \\
&\le \left\|(f+g)^{p-1}\right\|_{p/(p-1)} \|f\|_p + \left\|(f+g)^{p-1}\right\|_{p/(p-1)} \|g\|_p \\
&= \left(\int_X |f+g|^{(p-1)p/(p-1)} \, d\mu\right)^{(p-1)/p} \|f\|_p \\
&\quad + \left(\int_X |f+g|^{(p-1)p/(p-1)} \, d\mu\right)^{(p-1)/p} \|g\|_p \\
&= \|f+g\|_p^{p-1} \left(\|f\|_p + \|g\|_p\right),
\end{aligned}$$

hence

$$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

$\boxed{3}$ We are to prove the generalized Hölder inequality,

$$\left\|\prod_{j=1}^n f_j\right\|_q \le \prod_{j=1}^n \|f_j\|_{p_j}, \quad f_j \in L^{p_j}(X)$$

where

$$\sum_{j=1}^n \frac{1}{p_j} = \frac{1}{q}, \quad q, p_j \in [1,\infty].$$

First we note that if $p_j = \infty$ for some j, $j = n$ say, we have

$$\sum_{j=1}^n \frac{1}{p_j} = \sum_{j=1}^{n-1} \frac{1}{p_j} = \frac{1}{q},$$

and

$$\left\|\prod_{j=1}^n f_j\right\|_q \le \|f_n\|_\infty \left\|\prod_{j=1}^{n-1} f_j\right\|_q.$$

This we can do for any finite number of j-s where $p_j = \infty$ and reduce the problem to proving the inequality for $p_j \in [1,\infty)$. The only way to have $q = \infty$ is when $p_j = \infty$ for all j, but since

$$\|fg\|_\infty = \operatorname*{ess\,sup}_{x\in X} |f(x)||g(x)| \le \operatorname*{ess\,sup}_{x\in X} |f(x)| \operatorname*{ess\,sup}_{y\in X} |g(y)| = \|f\|_\infty \|g\|_\infty$$

we can easily iterate the above inequality to obtain the desired result in this setting.

Let us therefore consider $q, p_j \in [1,\infty)$. First we treat $n = 2$, with $1/p_1 + 1/p_2 = 1/q$, such that $q/p_1 + q/p_2 = 1$. Then from the standard Hölder inequality we get

$$\begin{aligned}
\|f_1 f_2\|_q^q &= \|f_1^q f_2^q\|_1 \le \|f_1^q\|_{p_1/q} \|f_2^q\|_{p_2/q} \\
&\le \left(\int_X |f_1|^{q(p_1/q)} \, d\mu\right)^{q/p_1} \left(\int_X |f_2|^{q(p_2/q)} \, d\mu\right)^{q/p_2} \\
&= \|f_1\|_{p_1}^q \|f_2\|_{p_2}^q,
\end{aligned}$$

and by taking the q-th root we obtain the inequality. For general n we get the result inductively. Let us assume that the inequality holds for $n = k$ and consider $n = k+1$. We define $g_j = f_j$, $r_j = p_j$ for $j = 1, \dots, k-1$, and $g_k = f_k f_{k+1}$, $r_k = p_k p_{k+1}/(p_k + p_{k+1})$ such that $1/r_k = 1/p_k + 1/p_{k+1}$. Note that analogous to the case $n = 2$ we have

$$\|g_k\|_{r_k}^{r_k} = \|f_k^{r_k} f_{k+1}^{r_k}\|_1 \le \|f_k^{r_k}\|_{p_k/r_k} \|f_{k+1}^{r_k}\|_{p_{k+1}/r_k} = \|f_k\|_{p_k}^{r_k} \|f_{k+1}\|_{p_{k+1}}^{r_k}$$

or

$$\|g_k\|_{r_k} \le \|f_k\|_{p_k} \|f_{k+1}\|_{p_{k+1}} < \infty,$$

hence $g_k \in L^{r_k}(X)$. Then our assumption is readily applicable:

$$\left\| \prod_{j=1}^{k+1} f_j \right\|_q = \left\| \prod_{j=1}^{k} g_j \right\|_q \le \prod_{j=1}^{k} \|g_j\|_{r_j} \le \prod_{j=1}^{k+1} \|f_j\|_{p_j},$$

and the result follows.

$\boxed{4}$ We are to prove

$$\|f * g\|_1 \le \|f\|_1 \|g\|_1$$

for $f, g \in L^1(\mathbb{R}^d, \lambda)$. By Tonelli's theorem it follows that

$$\begin{aligned} \|f * g\|_1 &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x-y) g(y)\, dy \right| dx \le \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)||g(y)|\, dy\, dx \\ &\underset{\text{Tonelli}}{=} \int_{\mathbb{R}^d} |g(y)| \int_{\mathbb{R}^d} |f(x-y)|\, dx\, dy = \|f\|_1 \int_{\mathbb{R}^d} |g(y)|\, dy = \|f\|_1 \|g\|_1. \end{aligned}$$

$\boxed{5}$ We are to prove Young's second convolution inequality

$$\|f(g * h)\|_1 \le \|f\|_p \|g\|_q \|h\|_r$$

for $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d), h \in L^r(\mathbb{R}^d)$ where $1/p + 1/q + 1/r = 2$ and $p, q, r \in [1, \infty]$. Here $\mu = \lambda$. First we recall Young's first convolution inequality,

$$\|g * h\|_s \le \|g\|_q \|h\|_r$$

for $g \in L^q(\mathbb{R}^d), h \in L^r(\mathbb{R}^d)$ where $1/q + 1/r = 1 + 1/s$ and $q, r, s \in [1, \infty]$. Then the Hölder inequality followed by the first Young inequality yields the result directly

$$\|f(g * h)\|_1 \le \|f\|_p \|g * h\|_{p/(p-1)} \le \|f\|_p \|g\|_q \|h\|_r,$$

as $1/p + 1/(p/(p-1)) = 1$ and $1/q + 1/r = 2 - 1/p = 1 + 1/(p/(p-1))$.