MA8105
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Solutions to exercise set 7

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1 Define the cut-off function ϕ_j and show that $\|f\phi_j\|_p \leq \|f\phi_j\|_p$, $\|f - f\phi_j\|_p \to 0$, $p \in [1, \infty)$.

Define the function ϕ to be 1 on $B(0,1)$ and 0 outside of $B(0,2)$ as in Exercise 0. Define the cutoff-function $\phi_j(x) = \phi(\frac{x}{j})$ and note, that ϕ_j equals 1 on $B(0,j)$, and vanishes outside of $B(0,2j)$.

Since $\phi_j(x) \leq 1$ $\forall x \in \mathbb{R}^d$, for any $f \in L^p$ we have

$$\|f\phi_j\|_p^p = \int_{\mathbb{R}^d} |f(x)\phi_j(x)|^p dx = \int_{\mathbb{R}^d} |f(x)|^p |\phi_j(x)|^p dx \leq \int_{\mathbb{R}^d} |f(x)|^p dx = \|f\|_p^p.$$

Since $f\chi_{B(0,j)} = f$ on $B(0,j)$, it follows that $f(1 - \chi_{B(0,j)}) \to 0$ a.e. and $|f(1 - \chi_{B(0,j)})| \leq |f|$. So we can apply Lebesgue's dominated convergence theorem to obtain:

$$\|f - f\phi_j\|_p^p = \int_{\|x\|>j} |f(x)(1 - \phi_j(x))|^p dx \to 0.$$

The L^∞ case

For $f \in L^\infty$ we have, $|f(x)\phi_j(x)| \leq |f(x)|$, so the following inequality also holds:

$$\|f\phi_j\|_\infty \leq \|f\|_\infty.$$

The function $f(x) = 1$ is an L^∞ function. We have $f(x)\phi_j(x) = 0$ for allmost all x with $\|x\| > 2j$. Therefore

$$\|f - f\phi_j\|_\infty \geq 1$$

for all j, since $\lambda(\mathbb{R}^d \setminus B(0,2j)) = \infty$. So we have no L^∞ convergence.

2 Let $\mathscr{F} \subseteq L^p(\mathbb{R}^d)$, $p \in [1, \infty)$, satisfy

1. $\sup_{f\in\mathscr{F}} \|f\|_p < \infty$

2. $\forall \epsilon > 0\ \exists \delta > 0 : \|y\| < \delta \Rightarrow \sup_{f\in\mathscr{F}} \|f - \tau_y f\|_p < \epsilon$.

Then $\{f|_\Omega | f \in \mathscr{F}\}$ is totally bounded in $L^p(\Omega)$ for a bounded set Ω.

Proof:

Let $R > 0$, such that $\Omega \subseteq B(0,R)$. Define $\hat{\mathscr{F}} = \{f\phi | f \in \mathscr{F}\}$, where ϕ is a cutoff function, which is 1 on Ω and vanisches outside of $B(0,2R)$. Then

1. $\hat{\mathscr{F}}$ is equibounded:
   $\sup_{\hat{f}\in\hat{\mathscr{F}}} \|\hat{f}\|_p \leq \sup_{f\in\mathscr{F}} \|f\|_p < \infty$ by assumption and Exercise 1.

2. $\hat{\mathscr{F}}$ is equicontinuous:
   ϕ is uniformely continuous as it is compactly supported and continuous. So $\forall \epsilon > 0\ \exists \hat{\delta} > 0$, such that $|y| < \hat{\delta} \Rightarrow |\phi(x) - \phi(x+y)| < \epsilon$. Then for a.e. $|y| < \hat{\delta}$,

$$\begin{aligned} |\hat{f}(x) - \tau_y \hat{f}(x)| &= |f(x)\phi(x) - f(x+y)\phi(x+y)| \\ &\leq |f(x)||\phi(x) - \phi(x+y)| + |f(x) - f(x+y)||\phi(x+y)| \\ &\leq \epsilon|f(x)| + |f(x) - \tau_y f(x)|. \end{aligned}$$

   Consequently: $\forall \epsilon > 0\ \exists \delta > 0:$
   $|y| < \min\{\delta, \hat{\delta}\} \Rightarrow \sup_{\hat{f}\in\hat{\mathscr{F}}} \|\hat{f} - \tau_y \hat{f}\|_p < (1 + \sup_{\hat{f}\in\hat{\mathscr{F}}} \|f\|_p)\epsilon$.

3. $\hat{\mathscr{F}}$ is tight:
   $\forall \hat{f} \in \hat{\mathscr{F}} : \int_{\|x\|>2R} |\hat{f}(x)|^p dx = 0$ by the definition of ϕ.

The family $\hat{\mathscr{F}}$ fullfils all prerequisites of the Kolmogorov-Riesz Theorem. Therefore $\hat{\mathscr{F}}$ is totally bounded in $L^p(\mathbb{R}^d)$.

Now, show, that the restriction to $L^p(\Omega)$ of an ϵ-net for $\hat{\mathscr{F}}$ is also an ϵ-net for $\{f|_\Omega \mid f \in \mathscr{F}\}$ in $L^p(\Omega)$.

Let $f \in \mathscr{F}$ and $\{\hat{f}_j\}_{j=1,\dots,N}$ be a ϵ-net for $\hat{\mathscr{F}}$.

Then $\phi f \in \hat{\mathscr{F}}$, $\exists k \in \{1, \dots, N\} : \|\phi f - \hat{f}_k\|_p \leq \epsilon$, and

$$\|f|_\Omega - \hat{f}_k|_\Omega\|^p_{L^p(\Omega)} = \int_\Omega |f(x) - \hat{f}_k(x)|^p dx \leq \|\phi f - \phi \hat{f}_k\|_p^p = \|\phi f - \hat{f}_k\|_p^p \leq \epsilon^p.$$

Consequently $\{f|_\Omega \mid f \in \mathscr{F}\}$ is totally bounded in $L^p(\Omega)$.

$\boxed{3}$ **Mass escaping to infinity: Let $f_k(x) = \chi_{[k,k+1]}(x)$.**

a)Show that $f_k \to 0$ pointwise, but that $\{f_k\}_k$ does not converge in $L^1(\mathbb{R})$ nor in measure.

For all $x \in \mathbb{R}$, there exists a $R \in \mathbb{N}$, such that $R > x$. For $\forall k > R$, we have $f_k(x) = 0$, so $\{f_k\}$ converges pointwise to 0.

Let's have a look at the L^1 norm of the f_k:

$$\|f_k\|_1 = \int_{\mathbb{R}^d} |f_k(x)| dx = \int_k^{k+1} 1 dx = 1,$$

so $\{f_k\}_k$ can not converge in norm to 0.

Show that $\{f_k\}_k$ does not converge in measure to 0:

$$\lambda(\{x \in \mathbb{R} | |f_k(x)| > \frac{1}{2}\}) = \lambda([k, k+1]) = 1,$$

and therefore $\exists \epsilon > 0 : \lim_{k\to\infty} \lambda(\{x \in \mathbb{R} | |f_k(x)| > \epsilon\}) \neq 0$.

b) Show that $f_k \to 0$ in $L^1_{loc}(\mathbb{R})$ and locally in measure.

Let $K \subseteq \mathbb{R}$ be a compact set. By the Heine-Borel theorem K is bounded. So for $k > \sup\{|x| \mid x \in K\}$, we have $f_k(x) = 0\ \forall x \in K$, and therefore $\|f_k\|_{L^1(K)} = 0$. So $\{f_k\}_k$ converges to 0 in $L^1_{loc}(\mathbb{R})$.

For a sequence f_k to converge locally in measure to f it has to fulfill:

$$\forall \epsilon > 0 : \lim_{k\to\infty} \mu(\{x \in K | |f_k(x) - f(x)| > \epsilon\}) = 0$$

for all compact sets K.

Let $\epsilon > 0$ and $K \subseteq \mathbb{R}$ an arbitrary compact set.

$$l_{k,\epsilon} := \lambda(\{x \in K | |f_k(x)| > \epsilon\}) \leq \lambda(K \cap [k, k+1])$$

As each compact set in $\mathbb{R}$ is bounded, $K \cap [k, k+1] = \emptyset$ for $k > \sup\{|x| \mid x \in K\}$, hence $\lim_{k\to\infty} l_{k,\epsilon} = 0$. Therefore $\{f_k\}_k$ converges locally in measure to 0.

c) Show, that $\{f_k\}_k$ is equibounded and -continuous in L^1. Is it tight?

As noted earlier $\|f_k\|_1 = 1$, so $\{f_k\}_k$ is obviously equibounded.

Equicontinuous: Let $-1 \leq y < 0$:

$$\|f_k(x) - f_k(x+y)\|_1 = \int_k^{k-y} 1dx + \int_{k+1}^{k+1-y} 1dx = 2|y| \quad \forall k$$

For $y < -1$:

$$\|f_k(x) - f_k(x+y)\|_1 = \int_k^{k+1} 1dx + \int_{k-y}^{k+1-y} 1dx = 2 \quad \forall k$$

Doing the same for positive y, we get:

$$\|f_k(x) - f_k(x+y)\|_1 \leq 2|y|$$

so the $\{f_k\}_k$ is equicontinuous when choosing $\delta = \frac{\epsilon}{2}$.

$\{f_k\}_k$ is not tight: For any $R > 0$, there exists a $k \in \mathbb{N}$ (take $k > R$), such that $\int_{|x|>R} f_k(x)dx = \int_k^{k+1} 1dx = 1$.

As we have seen earlier, the norm of the f_k is constant, so there can not be a subsequence converging to 0. So this sequence is an example, that the sequence being tight is a crucial prerequisite for the Kolmogorov-Riesz theorem.

Also $\lambda(\{x \in \mathbb{R} | |f_k(x)| > \frac{1}{2}\})$ is constant, so there can not be a subsequence converging to 0 in measure. But there is a subsequence (the sequence itself) converging to 0 locally in L^1 and locally in measure.

$\boxed{4}$ **Show that $\mathcal{F} := \{f(x) = \chi_{[a,b]}(x) | -1 < a < b < 1\}$ is totally bounded in L^p for $p \in [1, \infty)$.**

We use Kolmogorov-Riesz to show that $\mathcal{F}$ is totally bounded.

$\mathcal{F}$ is equibounded: $\|\chi_{[a,b]}\|_p = \left(\int_a^b 1dx\right)^{\frac{1}{p}} = (b-a)^{\frac{1}{p}} \leq 2^{\frac{1}{p}}$.

$\mathcal{F}$ is equicontinuous: As in Excercise 3, we have

$$\int_{\mathbb{R}^d} |\chi_{[a,b]}(x) - \chi_{[a,b]}(x+y)|^p dx \leq 2|y|.$$

$\mathcal{F}$ is tight: $\int_{|x|>1} \chi_{[a,b]} dx = 0$,

Therfore all prerequisites of Kolmogorov-Riesz are fulfilled and we can conclude, that $\mathcal{F}$ is totally bounded.

$\boxed{5}$ **Let $p \in [1,\infty)$ and $F \subseteq L^\infty(\Omega)$ where $\Omega \subseteq \mathbb{R}^d$ is open, bounded and measurable. Assume (C1) $\sup_f \|f\|_\infty < \infty$ and (C2) $\sup_f \|f - \tau_y f\|_{L^p((\Omega-y)\cap\Omega)} \to 0$ as $y \to 0$. Then F is totally bounded in $L^p(\Omega)$.**

Proof:

Define $\hat{f}(x) = \begin{cases} f(x) & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$. Show that, $\hat{F} = \{\hat{f} : f \in F\}$ fullfils the assumption of Kolmogorov-Riesz in $L^p(\mathbb{R}^d)$. Afterwards show, that $\hat{F}$ totally bounded in $L^p(\mathbb{R}^d)$ implies F totally bounded in $L^p(\Omega)$:

$\hat{F} \subseteq L^p(\mathbb{R}^d)$:

Since Ω is bounded and measurable, we know, that Ω has finite measure. So we can conclude:

$$\int_{\mathbb{R}^d} |\hat{f}(x)|^p dx = \int_\Omega |f(x)|^p dx \leq \int_\Omega \|f\|_\infty^p dx = \lambda(\Omega)\|f\|_\infty^p < \infty,$$

which holds for all $\hat{f} \in \hat{F}$.

$\hat{f}$ is equibounded in $L^p(\mathbb{R}^d)$:

Let $M = \sup_{f\in F} \|f\|_\infty$. We know from (C1), that $M < \infty$.

$$\begin{aligned} \sup_{\hat{f}\in\hat{F}} \|\hat{f}\|_p &= \sup_{\hat{f}\in\hat{F}} \left(\int_{\mathbb{R}^d} |\hat{f}(x)|^p dx \right)^{\frac{1}{p}} \\ &= \sup_{f\in F} \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}} \\ &\leq \sup_{f\in F} (\lambda(\Omega)\|f\|_\infty^p)^{\frac{1}{p}} \\ &\leq \lambda(\Omega)^{\frac{1}{p}} M < \infty \end{aligned}$$

$\hat{F}$ is equicontinuous in $L^p(\mathbb{R}^d)$:

For all $\hat{f} \in \hat{F}$, we have:

$$\begin{aligned} \|\hat{f} - \tau_y \hat{f}\|_p^p &= \int_{\mathbb{R}^d} |\hat{f}(x) - \hat{f}(x+y)|^p dx \\ &= \int_{\Omega\cup(\Omega-y)} |f(x) - f(x+y)|^p dx \\ &= \int_{\Omega\cap(\Omega-y)} (\dots) dx + \int_{\Omega\setminus(\Omega-y)} |f(x)|^p dx + \int_{(\Omega-y)\setminus\Omega} |f(x+y)|^p dx \\ &\leq \|f - \tau_y f\|_{L^p(\Omega\cap(\Omega-y))}^p + \lambda(\Omega\setminus(\Omega-y))\|f\|_\infty^p + \lambda((\Omega-y)\setminus\Omega)\|f\|_\infty^p \end{aligned}$$

By (C2) we have

$$\sup_{f\in F} \|f - \tau_y f\|^p_{L^p(\Omega\cap(\Omega-y))} \to 0$$

as $y \to 0$.

Also, we have

$$\lambda(\Omega \setminus (\Omega - y)), \lambda((\Omega - y) \setminus \Omega) \to 0$$

as $y \to 0$:

First note, that $\lambda(\Omega \setminus (\Omega - y)) = \int_{\mathbb{R}^d} \chi_{\Omega\setminus(\Omega-y)}(x)dx$. We use Lebesgue's dominated convergence theorem to show, that this integral goes to zero as y goes to 0.

$\chi_{\Omega\setminus(\Omega-y)}(x) \leq \chi_\Omega(x)$, which is a integrable function, since $\Omega \setminus (\Omega - y) \subseteq \Omega$.

Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^d$ converging to 0. Then for every $x \in \mathbb{R}^d \setminus \partial\Omega$, there exists an $N \in \mathbb{N}$, such that, $\chi_{\Omega\setminus(\Omega-y_n)}(x) = 0$ $\forall n > N$. To accomplish this, choose N, such that $\|y_n\| < dist(x, \partial\Omega)$ $\forall n > N$.

So $\chi_{\Omega\setminus(\Omega-y_n)} \to 0$ pointwise for all $x \in \mathbb{R}^d$. We can apply the dominated convergence theorem to obtain:

$$\lim_{n\to\infty} \lambda(\Omega \setminus (\Omega - y_n)) = \lim_{n\to\infty} \int \chi_{\Omega\setminus(\Omega-y_n)}(x)dx = \int \lim_{n\to\infty} \chi_{\Omega\setminus(\Omega-y_n)}(x)dx = 0,$$

and therefore $\lambda(\Omega \setminus (\Omega - y)) \to 0$ as $y \to 0$.

A similiar argument leads to $\lambda((\Omega - y) \setminus \Omega) \to 0$.

So we can conclude:

$$\begin{aligned}
&\sup_{\hat{f}\in\hat{F}} \|\hat{f} - \tau_y \hat{f}\|^p_p \\
&\leq \sup_{f\in F} \left(\|f - \tau_y f\|^p_{L^p(\Omega\cap(\Omega-y))} + \lambda(\Omega \setminus (\Omega - y))\|f\|^p_\infty + \lambda((\Omega - y) \setminus \Omega)\|f\|^p_\infty \right) \\
&= \sup_{f\in F} \|f - \tau_y f\|^p_{L^p(\Omega\cap(\Omega-y))} + \sup_{f\in F} \lambda(\Omega \setminus (\Omega - y))\|f\|^p_\infty + \sup_{f\in F} \lambda((\Omega - y) \setminus \Omega)\|f\|^p_\infty \\
&= \sup_{f\in F} \|f - \tau_y f\|^p_{L^p(\Omega\cap(\Omega-y))} + \lambda(\Omega \setminus (\Omega - y))M^p + \lambda((\Omega - y) \setminus \Omega)M^p \\
&\to 0
\end{aligned}$$

as $y \to 0$, which is a different statement for equicontinuity.

$\hat{F}$ is tight in $L^p(\Omega)$:

As Ω is bounded, we can find an $R > 0$, such that $\Omega \subseteq B(0, R)$. Then $\hat{f}(x) = 0$ for all $x \notin B(0, R)$.

Consequently:

$$\int_{\|x\|>R} |\hat{f}(x)|^p dx = 0$$

for all $\hat{f} \in \hat{F}$.

$\hat{F}$ is totally bounded in $L^p(\mathbb{R}^d)$:

As shown previously F fullfils all assumptions of Kolmogorov-Riesz, so we can conclude, that $\hat{F}$ is totally bounded in $L^p(\mathbb{R}^d)$.

F is totally bounded in $L^p(\Omega)$:

Let $\{\hat{f}_i\}_{i=1,\dots N}$ be an ϵ-net for $\hat{F}$ in $L^p(\mathbb{R}^d)$. We show, that $\{\hat{f}_i|_\Omega\}_{i=1,\dots N}$ is an ϵ-net for F in $L^p(\Omega)$.

Let $f \in F$ and $\hat{f} \in \hat{F}$, such that $\hat{f}|_\Omega = f$. As the $\hat{f}_i$ form an ϵ-net, there is a $k \in \{1, \dots N\}$ with $\|\hat{f} - \hat{f}_k\|_p < \epsilon$. Then:

$$\|f - f_k\|_{L^p(\Omega)}^p = \int_\Omega |f(x) - f_k(x)|^p dx \leq \int_{\mathbb{R}^d} |\hat{f}(x) - \hat{f}_k(x)|^p dx = \|\hat{f} - \hat{f}_k\|^p \leq \epsilon^p.$$