MA8105
Nonlinear PDEs and Sobolev spaces
Spring 2019

Exercise set 7

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1 Prove Lemma 20 from the lectures: Define the cut-off function ϕ_j and show that

$$\|f\phi_j\|_p \leq \|f\|_p, \quad \|f - f\phi_j\|_p \to 0, \quad p \in [0, \infty).$$

Explain also the L^∞-case.

2 Kolmogorov-Riesz 2, Theorem 27 in the lectures: L^p-equiboundedness and -equicontinuity of $\mathcal{F} \subset L^p$ implies total boundedness of the restriction of $\mathcal{F}$ to a bounded set Ω.

Formulate the precise result and prove it using Kolmogorov-Riesz 1 (the result in whole space, Theorem 26 in the lectures).

Hint: See hints in class – multiply $\mathcal{F}$ by a cut-off function which is 1 on Ω.

3 (Mass escaping to infinity) Let $f_k(x) = \chi_{[k,k+1]}(x)$.

a) Show that $f_k \to 0$ point wise, but $\{f_k\}_k$ does not converge in $L^1(\mathbb{R})$ or measure.

b) Show that $f_k \to 0$ in L^1_{loc} and locally in measure (only consider points in compact subsets of $\mathbb{R}$ – give a definition!).

c) Show that $\{f_k\}_k$ is equibounded and -continuous in L^1. Is it tight?
Prove that there is a convergent subsequence and explain what type of convergence we get.

4 Show that $\mathcal{F} := \left\{\chi_{[a,b]}(x) : -1 < a < b < 1\right\}$ is precompact in $L^p(\mathbb{R})$ for $p \in [1, \infty)$.

5 (Challenge) Let $p \in [1, \infty)$ and $\mathcal{F} \subset L^\infty(\Omega)$ where $\Omega \subset \mathbb{R}^d$ is bounded and open. Assume (C1) $\sup_f \|f\|_\infty < \infty$ and (C2) $\sup_f \|f - \tau_y f\|_{L^p((\Omega - y)\cap\Omega)} \to 0$ as $y \to 0$. Then $\mathcal{F}$ precompact in $L^p(\Omega)$.

Hint: Extend functions to $\mathbb{R}^d$ by 0 in Ω^c. Show that extension satisfies the conditions in the Kolmogorov-Riesz theorem (inspiration can be found in Exercice 4 ...).