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Solutions to exercise set 8

1 Counterexample for L^1

Define $g_n(x)4\chi_{(0,2\pi)}(x)(1+\sin(nx))$. We show that $g_n \rightharpoonup 4\chi_{(0,2\pi)}(x)$ in $L^1(\mathbb{R})$:
For $\varphi \in L^\infty(\mathbb{R})$,

$$|\langle \varphi, g_n - 4\chi_{(0,2\pi)}\rangle| = \left|\int_{\mathbb{R}} 4\varphi(x)\chi_{(0,2\pi)}(x)\sin(nx)dx\right| = 4\left|\int_0^{2\pi} \varphi(x)\sin(nx)dx\right|.$$

The simple functions are dense in $L^\infty(\mathbb{R})$. Let ψ_ϵ be a simple function such that $\|\varphi - \psi_\epsilon\| < \epsilon$. Then

$$\left|\int_0^{2\pi} \varphi(x)\sin(nx)dx\right| \leq \left|\int_0^{2\pi} \psi_\epsilon(x)\sin(nx)dx\right| + \left|\int_0^{2\pi} (\varphi(x)-\psi_\epsilon(x))\sin(nx)dx\right|.$$

For the last integral:

$$\left|\int_0^{2\pi} (\varphi(x)-\psi_\epsilon(x))\sin(nx)dx\right| \leq \|\varphi-\psi_\epsilon\|_\infty \|\sin(nx)\|_1 < 2\pi\epsilon.$$

For the first integral, we first consider $\psi_\epsilon = \chi_A$, where $A \subset (0,2\pi)$ is a measureable set. Since all simple functions are linear combinations of such indicator functions, it suffices to consider those. Since A is Lebesgue measureable, $\mu(A) < 2\pi$, we can find a compact subset $K \subset (0,2\pi)$ such that $\mu(K)+\delta > \mu((0,2\pi))$. Then K is covered by finitely many intervals of the form (a,b), $a,b \in [0,2\pi]$, and the integral reduces to

$$\begin{aligned}\left|\int_A \sin(nx)dx\right| &\leq \left|\int_{A\setminus K} \sin(nx)dx\right| + \left|\int_K \sin(nx)dx\right| \leq \mu(A\setminus K) + \sum_{i=1}^{m_\delta}\left|\int_{a_i}^{b_i} \sin(nx)dx\right| \\ &< \delta + \sum_{i=1}^{m_\delta} \frac{2}{n} \to 0\end{aligned}$$

From the above estimates, we see that $g_n \rightharpoonup 4\chi_{(0,2\pi)}(x)$ in $L^1(\mathbb{R})$.
Next, we show that $\|g_n\|_1 \to \|4\chi_{(0,2\pi)}\|_1$:

$$\begin{aligned}\left|\|g_n\|_1 - \|4\chi_{(0,2\pi)}\|_1\right| &= \left|\int_{\mathbb{R}} \left|4\chi_{(0,2\pi)}(x)(1+\sin(nx))\right|dx - \int_{\mathbb{R}}\left|4\chi_{(0,2\pi)}(x)\right|dx\right| \\ &= 4\left|\int_0^{2\pi}\sin(nx)dx\right| \xrightarrow{n\to\infty} 0.\end{aligned}$$

Finally, we show that $\|g_n - 4\chi_{(0,2\pi)}\|_1 = 16$:

$$\left|\|g_n - 4\chi_{(0,2\pi)}\|_1\right| = \int_{\mathbb{R}}\left|4\chi_{(0,2\pi)}(x)\sin(nx)\right|dx = 4\int_0^{2\pi}\left|\sin(nx)\right|dx = 16.$$

Counterexample for L^∞

Define $f_n = \chi_{(\frac{1}{n},1)}$. We show that $f_n {*} \chi_{(0,1)}$ in $L^\infty(\mathbb{R})$:

For $\varphi \in L^1(\mathbb{R})$,

$$|\langle f_n - \chi_{(0,1)}, \varphi\rangle| = \left| \int_{\mathbb{R}} \varphi(x)\chi_{(0,\frac{1}{n})}(x)dx \right|.$$

Here, $\varphi(x)\chi_{(0,\frac{1}{n})}(x) \to 0$ and $\left|\varphi(x)\chi_{(0,\frac{1}{n})}(x)\right| \leq \left|\varphi(x)\right|$, so by the dominated convergence theorem

$$\left| \int_{\mathbb{R}} \varphi(x)\chi_{(0,\frac{1}{n})}(x)dx \right| \xrightarrow{n\to\infty} 0,$$

and $f_n {*} \chi_{(0,1)}$ in $L^\infty(\mathbb{R})$.

That $\|f_n\|_\infty = \|\chi_{(0,1)}\|_\infty = 1$ is an immediate consequence of the definition of the essential supremum norm. Similarly, $\|f_n - \chi_{(0,1)}\| = \|\chi_{(0,\frac{1}{n})}\| = 1$, so we have our counterexample.

$\boxed{2}$ Let $f_n \in L^p(X)$ with $p \in (1,\infty)$. Then the following equivalence holds: $f_n \rightharpoonup f$ if and only if $\sup_n \|f_n\|_p < \infty$ and

$$\int_A f_n d\mu \to \int_A f d\mu$$

for any measurable set A, $\mu(A) < \infty$.

Proof. $f_n \rightharpoonup f$ means that $\langle \varphi, f_n\rangle \to \langle \varphi, f\rangle$ for any $\varphi \in L^q(X)$, where $\frac{1}{p} + \frac{1}{q} = 1$ such that $(L^q(X))^* = L^p(X)$. In other words,

$$\int_X \varphi(x) f_n(x) d\mu(x) \to \int_X \varphi(x) f(x) d\mu(x)$$

for any $\varphi \in L^q(X)$.

Suppose first that $f_n \rightharpoonup f$. Then have that $\sup_n \|f_n\|_p < \infty$ if and only if $\sup_n \langle f_n, \varphi\rangle < \infty$ for any $\varphi \in L^q(X)$ by the uniform boundedness principle, thus $\sup_n \|f_n\|_p < \infty$ is a necessary condition for $f_n \rightharpoonup f$. Moreover, since characteristic functions χ_A, for any measurable set A with $\mu(A) < \infty$, belong to $L^q(X)$ for any $q \in [1, \infty)$, we may take $\phi = \chi_A$ to see that

$$\int_X \chi_A(x) f_n(x) d\mu(x) \to \int_X \chi_A(x) f(x) d\mu(x),$$

equivalent to

$$\int_A f_n(x) d\mu(x) \to \int_A f(x) d\mu(x),$$

which is our our second claim.

Conversely, if $\sup_n \|f_n\|_p < \infty$ and

$$\int_A f_n(x) d\mu(x) \to \int_A f(x) d\mu(x)$$

for any measurable set A, $\mu(A) < \infty$, then we may use the density of simple functions in $L^q(X)$ to show that

$$\int_X \varphi(x) f_n(x) d\mu(x) \to \int_X \varphi(x) f(x) d\mu(x)$$

for any $\varphi \in L^q(X)$. Let ψ_ϵ be a simple function such that $\|\varphi - \psi_\epsilon\| < \epsilon$. Then

$$\begin{aligned}\left|\int_X \varphi(x)(f_n(x) - f(x)) d\mu(x)\right| &\leq \left|\int_X \psi_\epsilon(x)(f_n(x) - f(x)) d\mu(x)\right| + \left|\int_X (\varphi(x) - \psi_\epsilon(x))(f_n(x) - f(x)) d\mu\right. \\ &\leq \left|\int_X \Big(\sum_{i=1}^k a_i \chi_{A_i}(x)\Big)(f_n(x) - f(x)) d\mu(x)\right| + \|\varphi - \psi_\epsilon\|_q \|f_n - f\|_p \\ &< \sum_{i=1}^k a_i \left|\int_{A_i} (f_n(x) - f(x)) d\mu(x)\right| + \epsilon \sup_n \|f_n - f\|_p.\end{aligned}$$

Because

$$\int_{A_i} (f_n(x) - f(x)) d\mu(x) \to 0,$$

we can find N_i such that

$$\int_{A_i} (f_{N_i}(x) - f(x)) d\mu(x) < \frac{\epsilon}{a_i}.$$

Let then $N = \max_i\{N_i\}$. Then

$$\left|\int_X \varphi(x)(f_n(x) - f(x)) d\mu(x)\right| \leq \epsilon(1 + \sup_n \|f_n - f\|),$$

and $f_n \rightharpoonup f$. $\square$

3 Direct proof that $\{f_n\}$ is not uniformly integrable

We have the sequence $\{f_n\}$, $f_n(x) n \chi_{(0,\frac{1}{n})}(x)$. We show that there exists ϵ such that no $\delta > 0$ can guarantee

$$\sup_{n\in\mathbb{N}} \int_A |f_n| d\mu < \epsilon$$

for any $A \in X$, $\mu(A) < \delta$.

Consider $0 < \epsilon < 1$ and sets of the form $(0, \frac{1}{n})$. Then choosing n large enough guarantees that $\mu(A) = \frac{1}{n} < \delta$, yet

$$\sup_{n\in\mathbb{N}} \int_A |f_n| d\mu = 1 > \epsilon,$$

a contradiction to uniform integrability.

Indirect proof that $\{f_n\}$ is not uniformly integrable

We show that $\{f_n\}$ converges to δ_0 in distribution, i.e.

$$\int_{\mathbb{R}} f_n(x)\varphi(x)d\mu(x) \xrightarrow{n\to\infty} \varphi(0)$$

for any $\varphi \in C_c^\infty(\mathbb{R})$. Because any subsequence $\{f_{n_k}\}$ then also converges to δ_0 in distribution, and because $C_c^\infty(\mathbb{R}) \subset L^\infty(\mathbb{R})$, we can conclude that $\{f_n\}$ is not relatively weakly compact in $L^1(\mathbb{R})$. By Dunford-Pettis, this implies that $\{f_n\}$ is not uniformly integrable.

We have

$$\int_{\mathbb{R}} f_n(x)\varphi(x)d\mu(x) = n\int_0^{\frac{1}{n}} \varphi(x)d\mu(x) = n\varphi(x^*)\frac{1}{n} = \varphi(x^*)$$

for some $x^* \in (0, \frac{1}{n})$, by the mean value theorem. Letting n go to infinity, we obtain our result.

$\boxed{4}$ **Let $0 \neq g \in L^p(R^d), p \in (1,\infty)$, and $f_n(x) = g(x+n)$. Show what $f \rightharpoonup 0$ in $L^p(R^d)$.**

Weak convergence can be shown with thm 45. Since $g \in L^p(R^d), \forall\epsilon > 0 \exists N \in \mathbb{N}^d$ st $\forall n \geq N$ (componentwise), $\int_n^\infty |g|^p dx \leq \epsilon$.

Let $I = (a,b)$. Then $\forall n \geq N - a$, $\int_I f_n(x)d\mu \leq \epsilon^{1/p}\mu(I)^{1/q}$.

$\boxed{5}$ **Show the p=1 case in Example 4.25 (i) p 72 in Holden.**

First, note that since (0,1) is bounded, $L^\infty(0,1) \subset L^2(0,1)$. Also note that $\sup_n \|f_n\|_\infty = max(|a|,|b|)$, so $f_n \in L^\infty(0,1)$.

To check L^1 convergence of f_n, consider test functions $\varphi \in L^\infty(0,1)$. But then since both f_n and φ are in $L^2(0,1)$, $\int_0^1 f_n\varphi dx \to \int_0^1 f\varphi dx$ by applying the result for p=2.