Solutions to exercise set 9

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1 Show and interpret $f_n \to f$ in $L^1 \implies f_n \to f$ in $\mathcal{M}$.

Let $f_n, f \in L^1(X), d\mu_n = f_n dx, d\mu = f dx$, and $f_n \to f$ in L^1.

Recall that $\mu = \mu^+ - \mu^-$, where $\mu^+(E) = \sup \to_{A \subset E} \mu(A)$ and $\mu^-(E) = -\inf_{A\subset E} \mu(A)$. Similarly, $f = f^+ - f^-$, where $f^+(x) = \max(f(x), 0)$ and $f^-(x) = -\min(f(x), 0)$. Observe that if $d\mu = f dx$, then $d\mu^+ = f^+ d\mu$ and $d\mu^- = f^- d\mu$.

Then: $\|\mu_n - \mu\| = |\mu_n - \mu|(X) = ((\mu_n - \mu)^+ + (\mu_n - \mu)^-)(X) = \int_X d((\mu_n - \mu)^+ + (\mu_n - \mu)^-) = \int_X (f_n - f)^+ + (f_n - f)^- dx = \int_X |f_n - f| dx = \|f_n - f\|_L^1 \to 0$.

Interpretation: if a sequence of L^1 functions converges, then so do the measures associated with the functions.

2 Show Thm 74 $\implies$ Thm 75

Assumptions of Thm 75 implies there exists μ_{n_k}, μ Radon measures so that

$$\int g d\mu_{n_k} \to \int g d\mu \quad \forall g \in C_c. \tag{*}$$

Fix $f \in C_0$. Let $\epsilon > 0$ and $M = \max(\mu(X), \sup_k \mu_k(X)) < \infty$. Recall that C_0 is the closure of C_c wrt. $\|\cdot\|_\infty$, so $\exists f_n \in C_c$ st. $\|f - f_n\|_\infty \le \frac{\epsilon}{M}$. By (*), there is K so that $k \ge K$ implies $|\int f_n d(\mu_{n_k} - \mu)| \le \epsilon$.

Then: $|\int f d\mu - \int f d\mu_{n_k}| \le |\int f d\mu - \int f_n d\mu| + |\int f_n d\mu - \int f_n d\mu_{n_k}| + |\int f_n d\mu_{n_k} - \int f d\mu_{n_k}| \le \|f - f_n\|_\infty \mu(X) + \epsilon + \|f_n - f\|_\infty \sup_n \mu_n(X) \le 3\epsilon$

3 (a) $\implies$ (b) :

First, we assume that $f_n \to f$ in $W^{k,p}(K^\circ)$ for all $K \subset\subset \Omega$. Let $\varphi \in C_c^\infty(\Omega)$ and take $K \subset \Omega$ such that $\mathrm{supp}(\varphi) \subset K^\circ$. There is a $C > 0$ such that

$$\|\varphi\|_{L^\infty(\Omega)}, \|\partial^\alpha \varphi\|_{L^\infty(\Omega)} \le C \quad , \quad |\alpha| = 1.$$

By Hölder's inequality, we obtain

$$\|f\varphi\|^p_{L^p(\Omega)} = \|f^p \varphi^p\|_{L^1(\Omega)} \le \|f^p\|_{L^1(K^\circ)} \|\varphi^p\|_{L^\infty(K^\circ)} = C^p \|f\|^p_{L^p(K)}. \tag{1}$$

Applying this inequality above to the $W^{k,p}$-norm, we get

$$\begin{aligned}
&\|f_n\varphi - f\varphi\|^p_{W^{1,p}(\Omega)}\\
&= \sum_{|\alpha|\le 1} \|\partial^\alpha[(f_n - f)\varphi]\|^p_{L^p(\Omega)}\\
&= \|(f_n - f)\varphi\|^p_{L^p(\Omega)} + \sum_{|\alpha|=1}\Big(\|\varphi\partial^\alpha(f_n - f)\|^p_{L^p(\Omega)} + \|(f_n - f)\partial^\alpha\varphi\|^p_{L^p(\Omega)}\Big)\\
&\le C^p\|(f_n - f)\|^p_{L^p(K^\circ)} + \sum_{|\alpha|=1}\Big(C^p\|\partial^\alpha(f_n - f)\|^p_{L^p(K^\circ)} + C^p\|(f_n - f)\|^p_{L^p(K^\circ)}\Big)\\
&= (1+d)C^p\|(f_n - f)\|^p_{L^p(K^\circ)} + C^p|(f_n - f)|^p_{W^{1,p}(K^\circ)}\\
&\le (1+d)C^p\|(f_n - f)\|^p_{W^{1,p}(K^\circ)}
\end{aligned}$$

Thus, we have shown that $\|f_n\varphi - f\varphi\|_{W^{1,p}(\Omega)} \le \overline{C}\|(f_n - f)\|_{W^{1,p}(K^\circ)}$. If $n\to\infty$, then $\|f_n - f\|_{W^{1,p}(K^\circ)} \to 0$. This implies that $f_n\varphi \to f\varphi$ in $W^{k,p}(\Omega)$ for all $\varphi \in C_c^\infty(\Omega)$.

(b) $\implies$ (a) :
Now, we assume that $f_n\varphi \to f\varphi$ in $W^{k,p}(\Omega)$, which means that $\|f_n\varphi - f\varphi\|_{W^{k,p}} \to 0$. Let K be a set such that $(K^\circ) \subset \Omega$ and take a cut-off function $\varphi \in C_c^\infty$ such that $0 \le \varphi \le 1$ and $\varphi|_K \equiv 1$. Since $\mathbb{R}^d$ has bigger measure than K°, we have that

$$\|f_n - f\|_{W^{1,p}(K^\circ)} = \|f_n\varphi - f\varphi\|_{W^{1,p}(K^\circ)} \le \|f_n\varphi - f\varphi\|_{W^{1,p}(\mathbb{R}^d)}.$$

This holds because $\varphi|_K \equiv 1$. Thus, $\|f_n - f\|_{W^{1,p}(K^\circ)}$ is bounded above by a quantity which tends to zero as $n\to\infty$, and $f_n \to f$ in $W^{k,p}(K^\circ)$.

$\boxed{4}$ **1) $\|\cdot\|_{W^{k,p}}$ is a norm on $W^{k,p}$:**

- **Triangle inequality holds:**
Let $u, v \in W^{k,p}$ be arbitrary functions. By applying Minkowski's inequalities, both for integrals and sums, we get

$$\begin{aligned}
\|u+v\|_{W^{k,p}} &= \Bigg(\sum_{|\alpha|\le k}\|\partial^\alpha(u+v)\|^p_{L^p}\Bigg)^{1/p}\\
&= \|\,(\|\partial^\alpha(u+v)\|_{L^p})_{|\alpha|\le k}\,\|_{l^p}\\
&\le \Big\|(\|\partial^\alpha u\|_{L^p} + \|\partial^\alpha v\|_{L^p})_{|\alpha|\le k}\Big\|_{l^p}\\
&\le \Big\|(\|\partial^\alpha u\|_{L^p})_{|\alpha|\le k}\Big\|_{l^p} + \Big\|(\|\partial^\alpha v\|_{L^p})_{|\alpha|\le k}\Big\|_{l^p}\\
&\le \|u\|_{W^{k,p}} + \|v\|_{W^{k,p}}
\end{aligned}$$

- **Homogeneous norm:**
Let $\lambda \in \mathbb{R}$ and $u \in W^{k,p}$ be arbitrary. Then

$$\|\lambda u\|_{W^{k,p}} = \Bigg(\sum_{|\alpha|\le k}\|\partial^\alpha(\lambda u)\|^p_{L^p}\Bigg)^{1/p} = \Bigg(\sum_{|\alpha|\le k}|\lambda|^p\|\partial^\alpha u\|^p_{L^p}\Bigg)^{1/p} = |\lambda|\|u\|_{W^{k,p}}$$

- **Zero element is unique:**
If $u = 0$, then $\|\partial^\alpha u\|_{L^p} = 0$ for all $|\alpha| \leq k$. Now, we just assume that $\|u\|_{W^{k,p}} = 0$. If $u \neq 0$, then $\|u\|_{L^p} > 0$, and since this quantity is a part of the $W^{k,p}$-norm, it will make $\|u\|_{W^{k,p}} > 0$, which is a contradiction. Thus, the zero element is unique.

2) $W^{k,p}$ is a vector space:
Assume that $u, v \in W^{k,p}$ and $a, b \in \mathbb{R}$. Then, by the properties of the norm,

$$\|au + bv\|_{W^{k,p}} \leq |a|\|u\|_{W^{k,p}} + |b|\|v\|_{W^{k,p}}$$

Thus, $au + bv \in W^{k,p}$, and $W^{k,p}$ is closed under addition and scalar multiplication, i.e. $W^{k,p}$ a vector space.

3) $W^{k,p}$ is a Banach space:
If $\{u_n\}_{n=1}^\infty$ is an arbitrary Cauchy sequence in $W^{k,p}$, so does $\{\partial^\alpha u_n\}_{n=1}^\infty$ in L^p for all $|\alpha| \leq k$, since ∂^α is linear. Since L^p is complete, there are u and u_α such that

$$n \to \infty \quad \Longrightarrow \quad \begin{cases} u_n \to u & \text{in } L^p \\ \partial^\alpha u_n \to u_\alpha & \text{in } L^p \end{cases}$$

Since $L^p \subset L^p_{\text{loc}}$, u_n defines a distribution $T_{u_n} \in \mathcal{D}'$. If $\varphi \in \mathcal{D}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$|T_{u_n}(\varphi) - T_u(\varphi)| \leq \int |u_n - u||\varphi| \, dx \leq \|u_n - u\|_{L^p}\|\varphi\|_{L^q}$$

Thus, $T_{u_n}\varphi \to T_u\varphi$ for all $\varphi \in \mathcal{D}'$. Similarly, we get

$$T_{u_\alpha}(\varphi) = \lim_{n\to\infty} T_{\partial^\alpha u_n}(\varphi) = \lim_{n\to\infty}(-1)^{|\alpha|}T_{u_n}(\partial^\alpha\varphi) = (-1)^{|\alpha|}T_u(\partial^\alpha\varphi)$$

Since $\varphi \in \mathcal{D}$, so is $\partial^\alpha\varphi$, and $u_\alpha = \partial^\alpha u$ in the distributional sense when $u \in W^{k,p}$. Since $\|u_n - u\|_{W^{k,p}} \to 0$, $W^{k,p}$ is a Banach space.

$\boxed{5}$ Let $f \in W^{m,p}(\Omega)$ and $\varphi \in C_b^m(\Omega)$ such that $\Omega \subset \mathbb{R}^d$. To derive the estimate, we use the product rule, Minkowski's inequality, Hölder's generalized inequality, and the continuous embedding $W^{m,p}(\Omega) \hookrightarrow L^p(\Omega)$:

$$\begin{aligned}
\|f\varphi\|_{W^{m,p}}^p &= \sum_{|\alpha|\le m} \|\partial^\alpha(f\varphi)\|_{L^p}^p \\
&= \sum_{|\alpha|\le m} \left\| \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \left(\partial^{\alpha-\beta} f \partial^\beta\varphi\right)\right\|_{L^p}^p \\
&\le \sum_{|\alpha|\le m} \left(\sum_{\beta\le\alpha} \binom{\alpha}{\beta} \|\partial^{\alpha-\beta} f \partial^\beta\varphi\|_{L^p}\right)^p \\
&\le \sum_{|\alpha|\le m} \left(\sum_{\beta\le\alpha} \binom{\alpha}{\beta} \|\partial^{\alpha-\beta} f\|_{L^p} \|\partial^\beta\varphi\|_{L^\infty}\right)^p \\
&\le \sum_{|\alpha|\le m} \left(\sum_{\beta\le\alpha} \binom{\alpha}{\beta} \|f\|_{W^{m,p}} \|\varphi\|_{W^{m,\infty}}\right)^p \\
&= \underbrace{\left[\sum_{|\alpha|\le m} \left(\sum_{\beta\le\alpha} \binom{\alpha}{\beta} \right)^p \right]}_{C^p} \|f\|_{W^{m,p}}^p \|\varphi\|_{W^{m,\infty}}^p
\end{aligned}$$

If we take the p-root on both sides of the inequality, we obtain

$$\|f\varphi\|_{W^{m,p}} \le C\|\varphi\|_{W^{m,\infty}}\|f\|_{W^{m,p}} \tag{2}$$