MA8109 Stokastiske prosesser i systemteori
(Stochastic Differential Equations)
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Exam Questions with Solutions

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Problem 1

(a) *Define the standard one-dimensional Brownian motion B_t starting at $x = 0$, and compute $\mathsf{E}(B_t^2)$ and $\operatorname{Var}(B_t^2)$.*

Let $\mathcal{P}$ be a partition of the interval $[0,t]$ so that $0 = t_0 < t_1 < \cdots < t_n = t$, $\Delta_k = t_{k+1} - t_k$, and $\Delta B_k = B_{t_{k+1}} - B_{t_k}$. Consider the process Y_t defined by $Y_t = \lim_{\mathcal{P}\to 0} \sum_{\mathcal{P}} (\Delta B_k)^2$ (limit in $L^2(\Omega)$).

(b) *Show that $Y_t = t$ a.s. (e.g. by computing $\mathsf{E}Y_t$ and Y_t).*

Solution:

(a) The axioms:

$$\begin{array}{ll} (i) & B_t \text{ is a Gaussian process for } t \geq 0 \text{, starting } x = 0, \\ (ii) & \mathsf{E}B_t = 0, \\ (iii) & \operatorname{Cov}(B_t, B_s) = \min(s,t). \end{array} \tag{1}$$

By applying the axioms and the formula for $\mathsf{E}B_t^4$ in the table, we have

$$\mathsf{E}\left(B_t^2\right) = \operatorname{Var}(B_t) = t, \tag{2}$$

$$\operatorname{Var}\left(B_t^2\right) = \mathsf{E}\left(B_t^2 - t\right)^2 = \mathsf{E}\left(B_t^4 - 2tB_t^2 + t^2\right) = 3t^2 - 2t \times t + t^2 = 2t^2. \tag{3}$$

(b) We first observe that

$$\mathsf{E}\left(\sum_{\mathcal{P}} (\Delta B_k)^2\right) = \sum_{\mathcal{P}} \mathsf{E}(\Delta B_k)^2 = \sum_{\mathcal{P}} \Delta_k = t \tag{4}$$

for all partitions, and therefore, $\mathsf{E}Y_t = t$. Moreover, since ΔB_k and ΔB_l are independent for $k \neq l$,

$$\begin{aligned} \operatorname{Var}\left(\sum_{\mathcal{P}} (\Delta B_k)^2\right) &= \sum_{\mathcal{P}} \operatorname{Var}\left((\Delta B_k)^2\right) = \sum_{\mathcal{P}} 2\Delta_k^2 \\ &\leq 2\max|\Delta_k| \sum_{\mathcal{P}} \Delta_k = 2\max|\Delta_k|\, t \underset{\mathcal{P}\to 0}{\longrightarrow} 0. \end{aligned} \tag{5}$$

Hence $\operatorname{Var} Y_t = 0$, and Y_t is equal to t a.s.

Alternatively, we could write, applying that independence implies orthogonality in $L^2(\Omega)$:

$$\begin{aligned}\left\| \sum_{\mathcal{P}} (\Delta B_k)^2 - t \right\|^2_{L^2(\Omega)} &= \sum_{\mathcal{P}} \mathsf{E}\left((\Delta B_k)^2 - \Delta_k \right)^2 \\ &= \sum_{\mathcal{P}} 2\Delta_k^2 \leq 2 \max |\Delta_k| \, t \underset{\mathcal{P}\to 0}{\longrightarrow} 0.\end{aligned}$$

Problem 2

Show that $B_t^2/t \in \mathcal{V}[0,T]$, $T < \infty$, and state the expectation and variance of

$$I = \int_0^T \frac{B_t^2}{t} dB_t. \tag{6}$$

Solution:

The function B_t^2/t is clearly $\mathcal{F}_t$-adapted (and $[0,T] \times \Omega$ measurable). Moreover,

$$\int_0^T \int_\Omega \left| \frac{B_t^2}{t} \right|^2 dP(\omega)\, dt = \int_0^T \frac{3t^2}{t^2} dt = 3T < \infty, \tag{7}$$

showing that $B_t^2/t \in \mathcal{V}[0,T]$.

We have $\mathsf{E}I = 0$ for all Itô Integrals, whereas $\operatorname{Var} I$ is equal to the integral in Eq. 7 by the Itô Isometry.

Problem 3

Assume that the regular (non-random) function θ_t is in $L^2[0,T]$, and consider the one-dimensional Itô process

$$X_t = -\int_0^t \frac{\theta_s^2}{2} ds + \int_0^t \theta_s dB_s, \; t \in [0,T]. \tag{8}$$

Let $M_t = \exp X_t$.

(a) *Compute the mean and variance of M_t by observing that $Y_t = \int_0^t \theta_s dB_s$ is Gaussian.*

(b) *Derive the stochastic differential equation for M_t and explain why M_t should be a Martingale.*

(c) *Verify from the definition of a Martingale that M_t is an $L^2(\Omega)$-Martingale with respect to $\mathcal{F}_t$ (The filtration of the Brownian motion).*

Solution:

(a) We first observe that $\mathsf{E}Y_t = 0$ and $\operatorname{Var} Y_t = \int_0^t \theta_s^2 ds$ (Itô Isometry). Then, since

$$M_t = \exp\left(-\int_0^t \frac{\theta_s^2}{2} ds \right) \times \exp Y_t, \tag{9}$$

and using the formula in the list for e^{Y_t}:

$$\mathsf{E}M_t = \exp\left(-\int_0^t \frac{\theta_s^2}{2}ds\right) \times \exp\left(\frac{1}{2}\operatorname{Var} Y_t\right) = 1,$$

$$\operatorname{Var} M_t = \mathsf{E}M_t^2 - 1 \ = \exp\left(-\int_0^t \theta_s^2 ds\right)\exp\left(\frac{1}{2}\operatorname{Var}(2Y_t)\right) - 1 = \exp\left(\int_0^t \theta_s^2 ds\right) - 1. \tag{10}$$

(b) We apply Itô's Formula:

$$\begin{aligned} dM_t &= (\exp X_t)\, dX_t + \frac{1}{2}(\exp X_t)(dX_t)^2 \\ &= M_t\left(-\frac{\theta_t^2}{2}dt + \theta_t dB_t\right) + \frac{1}{2}M_t\theta_t^2 dt = M_t\theta_t dB_t. \end{aligned} \tag{11}$$

Hence,

$$M_t - M_0 = \int_0^t M_s\theta_s dB_s. \tag{12}$$

The Itô integral is an $\mathcal{F}_t$-martingale w.r.t. its upper limit, and so is therefore M_t (in addition, M_t is also an martingale with respect to its own filtration).

(c)

1. Since θ_t is a regular deterministic function, X_t is clearly $\mathcal{F}_t$-measurable, and so is therefore also $M_t = \exp X_t$.
2. Since $\operatorname{Var} M_t$ is finite, $M_t \in L^2(\Omega) \subset L^1(\Omega)$.
3. For $0 \le t < s \le T$ we have:

$$\begin{aligned} \mathsf{E}(M_s|\mathcal{F}_t) &= \mathsf{E}\left(\exp\left(\int_0^s -\frac{\theta_u^2}{2}du + \theta_u dB_u\right)\middle|\, \mathcal{F}_t\right) \\ &= \mathsf{E}\left(M_t \exp\left(\int_r^s -\frac{\theta_u^2}{2}du + \theta_u dB_u\right)\middle|\, \mathcal{F}_t\right) \\ &= M_t\mathsf{E}\left(\exp\int_r^s \left\{-\frac{\theta_u^2}{2}du + \theta_u dB_u\right\}\right) \\ &= M_t. \end{aligned} \tag{13}$$

The last equalities follow since M_t is $\mathcal{F}_t$-measurable, whereas $\exp\int_t^s \left\{-\frac{\theta_u^2}{2}du + \theta_u dB_u\right\}$ is independent of $\mathcal{F}_t$.

Problem 4

Consider the stochastic process $X_t = \log(B_t)$, $X_0 = 0$ $(B_0 = 1)$. Write X_t as an autonome Itô diffusion. Does this differential equation satisfy the sufficient conditions for existence of solutions on an interval $[0, T]$?

Solution:

Clearly, since nothing prevents B_t from becoming negative, there will *always* be a fraction of the paths of X_t blowing up for a $t \in (0, T]$, regardless the size of $T > 0$.

The equation for X_t follows from Itô's Formula:

$$dX_t = \frac{1}{B_t} dB_t - \frac{1}{B_t^2} dt \;\; = -e^{-2X_t} dt + e^{-X_t} dB_t. \tag{14}$$

For large *negative* values of X_t (which may well occur), no bound like

$$\left| e^{-x} \right| \leq C \left(1 + |x|\right) \tag{15}$$

will work (B.Ø. Thm. 5.2.1).

Problem 5

Solve the equation

$$dX_t = -2tX_t dt + e^{-t^2} B_t dB_t, \;\; X_0 = 1, \;\; t \geq 0. \tag{16}$$

Solution:

We multiply through with $h(t)$ and replace $h(t)\, dX_t$ by $d[h(t) X_t] - h'(t) X_t dt$:

$$d[h(t) X_t] - h'(t) X_t dt = -h(t) 2tX_t dt + h(t) e^{-t^2} B_t dB_t. \tag{17}$$

The smart choice is clearly

$$h'(t) = 2th(t), \tag{18}$$

with a solution $h(t) = e^{t^2}$. The equation is now reduced to

$$d\left(e^{t^2} X_t\right) = B_t dB_t, \tag{19}$$

which may be integrated to

$$e^{t^2} X_t = X_0 + \int_0^t B_s dB_s. \tag{20}$$

The Itô integral is solvable by observing that with $Y_t = B_t^2$, we obtain from Itô's Formula

$$dY_t = 2B_t dB_t + \frac{1}{2} 2dt, \tag{21}$$

from which it follows that

$$\int_0^t B_s dB_s = \frac{1}{2}(Y_t - Y_0) - \int_0^t ds = \frac{B_t^2}{2} - \frac{t}{2}, \tag{22}$$

and finally, with $X_0 = 1$,

$$X_t = e^{-t^2} \left(1 + \int_0^t B_s dB_s\right) = e^{-t^2} \left(1 + \frac{1}{2}\left(B_t^2 - t\right)\right). \tag{23}$$

Problem 6

(a) *Dynkin's Formula may be stated*

$$\mathsf{E}^x f(X_\tau) = f(x) + \mathsf{E}^x \int_0^\tau Af(X_s)\,ds. \tag{24}$$

Explain the terms in the formula and how it is applied for solving the equation $Af = 0$.

For (b) and (c) we assume known that the average first exit time for Brownian motion is finite for all bounded domains.

(b) *Consider a domain in $\mathbb{R}^2$ bounded by two concentric circles,*

$$U = \left\{x \in \mathbb{R}^2;\ 0 < r < |x| < R < \infty\right\}. \tag{25}$$

A Brownian motion starts at $x \in U$. Compute the expectation of the exit time τ_U^x and the probabilities that the Brownian motion first exits through the inner and outer circle, respectively ($\mathsf{E}\tau_U^x$ is finite for all finite domains). What happens if we let $R \to \infty$?

(c) *Consider a Brownian motion in $\mathbb{R}^n$, $n \geq 3$, starting at x and let S be a sphere with radius $R > 0$ not containing x. Compute the average of the first hitting time of the sphere.*

Solution

(a)

- τ is a stopping time where we know that $\mathsf{E}^x\tau < \infty$ at all x-s we need.
- X_t is an Itô Diffusion, $dX_t = \beta_i(X_t)\,dt + \sigma(X_t)\,dB_t$, where β and σ fulfill the conditions in B.Ø. Thm. 5.2.1.
- A is the generator for the diffusion (stated in the formula list).
- $f \in C_0^2(\mathbb{R}^n)$.

If we are seeking a solution $Af(x) = 0$ at x in a set U, we let τ_U^x be the first exit time from U and consider

$$f(x) = \mathsf{E}^x f(X_{\tau_U}) \tag{26}$$

to be a candidate for the solution at x. This is true for "nice" problems.

(b) The generator for the Brownian motion is $A = \frac{1}{2}\nabla^2$. Let p_R be the probability that B_t^x exits for the first time through the outer circle (and $p_r = 1 - p_R$ for first exit through the inner circle). We apply Dynkin Formula with functions which are equal to $f_1(x) = \log|x|$ and $f_2(x) = |x|^2$ for $x \in U$. Outside U we assume that the functions are adjusted so that they belong to $C_0^2(\mathbb{R}^2)$ (or even $C_c^2(\mathbb{R}^2)$).

Since $Af_1(x) \equiv 0$ in U, we have

$$\mathsf{E}^x f_1(X_\tau) = p_R \log R + (1 - p_R)\log r = \log|x|, \tag{27}$$

and

$$p_R = \frac{\log|x| - \log r}{\log R - \log r}, \tag{28}$$

$$p_r = 1 - p_R = \frac{\log R - \log|x|}{\log R - \log r}. \tag{29}$$

We then apply f_2, and observe first that

$$Af_2(x) = \frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\left(x_1^2 + x_2^2\right) = 2. \tag{30}$$

Then

$$\mathsf{E}^x f_2(X_\tau) = p_R R^2 + (1 - p_R) r^2 = |x|^2 + 2 \times \mathsf{E}^x \tau_U \tag{31}$$

and

$$\mathsf{E}^x \tau_U = \left[p_R R^2 + (1 - p_R) r^2 - |x|^2\right]/2 \tag{32}$$

(A direct proof that the right hand side is indeed larger than 0 for $r < |x| < R$ is left to the reader!). When $R \to \infty$, then $p_r \to 1$ and $p_R \to 0$. However, it is clear that $\mathsf{E}^x \tau_U \to \infty$, since $p_R R^2 \sim R^2/\log R \underset{R\to\infty}{\longrightarrow} \infty$.

(c) This point starts similar to (b), applying $f_1(x) = |x|^{2-n}$:

$$p_R R^{2-n} + (1 - p_R) r^{2-n} = |x|^{2-n}. \tag{33}$$

Thus,

$$p_R = \frac{|x|^{-n+2} - r^{2-n}}{R^{-n+2} - r^{2-n}},$$

$$p_r = \frac{R^{2-n} - |x|^{2-n}}{R^{2-n} - r^{2-n}}. \tag{34}$$

Since $R^{-n+2} \to 0$ when $R \to \infty$, we have

$$\lim_{R\to\infty} p_r = \left(\frac{r}{|x|}\right)^{n-2},$$

$$\lim_{R\to\infty} p_R = 1 - (r/|x|)^{n-2}. \tag{35}$$

The first hitting time of the inner sphere is thus ∞ for a strictly positive fraction, $1 - (r/|x|)^{n-2}$, of the paths (which never hit S). This implies that $\mathsf{E}\tau_S^x$ must be infinite.

List of useful formulae

Note: The list does not state requirements for the formulae to be valid.

1D Gaussian variable $X \in \mathcal{N}(\mu, \sigma^2)$;

$$\mathsf{E}(X-\mu)^4 = 3\sigma^4, \tag{36}$$

$$\mathsf{E}\left(e^{X-\mu}\right) = e^{\frac{\sigma^2}{2}}. \tag{37}$$

Two formulae for Conditional Expectations:

$$\begin{array}{ll} (i) & \text{If } Y \text{ is } \mathcal{H}\text{-measurable, then } \mathsf{E}(YX|\mathcal{H}) = Y\mathsf{E}(X|\mathcal{H}). \\ (ii) & \text{If } X \text{ is independent of } \mathcal{H}\text{, then } \mathsf{E}(X|\mathcal{H}) = \mathsf{E}(X). \end{array} \tag{38}$$

The Itô Isometry:

$$\mathsf{E}\left|\int_0^T f(t,\omega)\,dB_t(\omega)\right|^2 = \int_0^T \mathsf{E}\,|f(t,\omega)|^2\,dt = \|f\|^2_{L^2(\Omega\times[0,T])} \tag{39}$$

Itô Formula:

$$dg(t,X_t,Y_t) = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial x}dX_t + \frac{\partial g}{\partial y}dY_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dX_t)^2 + \frac{\partial^2 g}{\partial x\partial y}dX_t dY_t + \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(dY_t)^2. \tag{40}$$

and *Rules*.

The Generator for $dX_t = \beta_i(X_t)\,dt + \sigma(X_t)\,dB_t$:

$$A(f)(x) = \sum_{i=1}^n \beta_i(x)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^n \left(\sigma(x)\,\sigma(x)'\right)_{i,j}\frac{\partial^2 f}{\partial x_i \partial x_j}(x). \tag{41}$$

Potential Solutions:

$$\begin{aligned} &\nabla^2 f = 0 \text{ for all } x \in \mathbb{R}^n,\ |x| \neq 0: \\ &\quad n = 2:\ \ f(x) = \log(|x|), \\ &\quad n > 2:\ \ f(x) = |x|^{2-n}. \end{aligned} \tag{42}$$