MA 8101 Stokastiske metoder i systemteori

AUTUMN TERM 2003

Suggested solution with some extra comments

The exam had a list of useful formulae attached. This list has been added here as well.

1 Problem

In this problem we are considering a standard Brownian motion B_t in $\mathbb{R}^1$ starting at 0.

(a) State the basic properties of the Brownian motion. Define, for a fixed $a > 0$, the process

$$X_{\ t} = aB_{t/a^2}. \tag{1}$$

Verify that also X_t is a standard Brownian motion.

A standard Brownian motion starting at 0 is a *Gaussian stochastic process* defined for $t \in [0, \infty)$ fulfilling

1. $\mathsf{E}B_t = 0,$
2. $\mathrm{Cov}\,(B_t B_s) = \min\,(t, s)\,.$

From (2) it follows that a B.M. has orthogonal increments. There exists a version of B.M. with continuous paths.

It is obvious that X_t is a Gaussian process (This actually requires that all finite collections $(X_{t_1}, \cdots, X_{t_N})$ are multivariate Gaussian, but this follows since B_t has such a property). Moreover, 1. clearly true. Finally,

$$\begin{aligned} \mathrm{Cov}\,(X_t, X_s) &= a^2\,\mathrm{Cov}\left(B_{s/a^2} B_{t/a^2}\right) \\ &= a^2 \times \min\left(\frac{s}{a^2}, \frac{t}{a^2}\right) \\ &= \min\,(s, t)\,. \end{aligned} \tag{2}$$

(b) *Let $0 = t_0 < t_1 < \cdots < t_{N+1} = T$ be a partition of the interval $[0, T]$ and φ the elementary function*

$$\varphi\,(t, \omega) = \sum_{j=0}^{N} e_j\,(\omega)\,\chi_{[t_{j+1} - t_j)}\,(t)\,. \tag{3}$$

What does it mean that φ is in the class $\mathcal{V}[0, T]$, and what is then the value of the Itô integral

$$\int_0^T \varphi\,(t, \omega)\,dB_t\,(\omega)? \tag{4}$$

Show that

$$\mathsf{E}\left(\int_0^T f\,(t, \omega)\,dB_t\,(\omega)\right) = 0. \tag{5}$$

for all $f \in \mathcal{V}[0, T]$.

The class $\mathcal{V}[0,T]$ consists of $\mathcal{B}\times\mathcal{F}$-measurable functions $f(t,\omega)\in L^2(\Omega\times[0,T])$ such that $f(t,\omega)$ is $\mathcal{F}_t$-measurable for all $t\in[0,T]$. Here this will be the case if $\mathsf{E}\left(e_j^2\right)<\infty$ and e_j is $\mathcal{F}_{t_j}$-measurable for all $j=0,\cdots,N$. Also,

$$\int_0^T \varphi(t,\omega)\,dB_t(\omega)=\sum_{j=0}^{N} e_j(\omega)\left[B_{t_{j+1}}(\omega)-B_{t_j}(\omega)\right]. \tag{6}$$

Since e_j and $B_{t_{j+1}}-B_{t_j}$ are independent,

$$\mathsf{E}\left(e_j\left[B_{t_{j+1}}-B_{t_j}\right]\right)=\mathsf{E}\left(e_j\right)\mathsf{E}\left(B_{t_{j+1}}-B_{t_j}\right)=0. \tag{7}$$

Thus,

$$\mathsf{E}\left(\int_0^T \varphi(t,\omega)\,dB_t(\omega)\right)=0. \tag{8}$$

In general, the Itô integral is a limit of integrals of simple functions. This is about all we requite for the exam, but the full argument is as follows: We find a sequence $\{\varphi_n\}$ such that $\mathsf{E}\left(\left|\int\varphi_n dB-\int f dB\right|^2\right)\underset{n\to\infty}{\longrightarrow}0$. Then

$$\begin{aligned}\left|\mathsf{E}\left(\int f dB\right)\right| &= \\ \left|\mathsf{E}\left(\int\varphi_n dB\right)-\mathsf{E}\left(\int f dB\right)\right| &= \left|\mathsf{E}\int(\varphi_n-f)\,dB\right| \\ &\le \mathsf{E}\left|\int(\varphi_n-f)\,dB\right| \\ &\le \left(\mathsf{E}\left(\left|\int(\varphi_n-f)\,dB\right|^2\right)\right)^{1/2}\underset{n\to\infty}{\longrightarrow}0.\end{aligned} \tag{9}$$

(c) Compute the variance of the integral

$$\int_0^1 tB_t(\omega)\,dB_t(\omega). \tag{10}$$

This follows immediately from Itô's Isometry since we know that the expectation is 0:

$$\begin{aligned}\mathrm{Var}\left(\int_0^1 tB_t(\omega)\,dB_t(\omega)\right) &= \mathsf{E}\left(\int_0^1 tB_t(\omega)\,dB_t(\omega)\right)^2 \\ &= \int_0^1 \mathsf{E}\,(tB_t)^2\,dt=\int_0^1 t^2\cdot t\,dt=\frac{1}{4}.\end{aligned} \tag{11}$$

2 Problem

We consider two Itô processes X_t and Y_t on R^1.

(a) Let X_t and Y_t be two Itô processes X_t and Y_t on $\mathbb{R}^1$. Prove that

$$d(X_tY_t)=X_t dY_t+Y_t dX_t+dX_t dY_t. \tag{12}$$

For this formula we apply the 2D Itô formula for the function $g(x,y) = xy$.Then

$$\begin{aligned} d(X_tY_t) &= \frac{\partial g}{\partial x} dX_t + \frac{\partial g}{\partial y} dY_t + \frac{\partial^2 g}{\partial x \partial y} dX_t \cdot dY_t \\ &= Y_t dX_t + X_t dY_t + 1 \cdot dX_t \cdot dY_t. \end{aligned} \tag{13}$$

(b) Let

$$X_t = e^{t/2} \sin(B_t). \tag{14}$$

Show that X_t can be written as an Itô integral.

We compute dX_t using Ito's formula:

$$\begin{aligned} dX_t &= \frac{1}{2} X_t dt + e^{t/2} \cos(B_t) dB_t + \frac{1}{2} e^{t/2} (-\sin B_t) dt \\ &= e^{t/2} \cos(B_t) dB_t. \end{aligned} \tag{15}$$

Hence,

$$X_t = \int_0^t e^{s/2} \cos(B_s) dB_s, \tag{16}$$

since it is clear that $e^{t/2} \cos(B_t) \in \mathcal{V}[0,T]$.

(c) The conclusion in (2.b) implies that X_t is a Martingale with respect to the filtration of the Brownian motion, $\mathcal{F}_t$. Prove this directly by applying the definition of a Martingale to the expression for X_t in Eqn. 14.

The first is to observe that

$$\mathsf{E}(X_t) \leq \mathsf{E}(|X_t|) = \mathsf{E}\left(e^{t/2} |\sin(B_t)|\right) \leq e^{t/2} < \infty. \tag{17}$$

Since X_t is a determininistic, continuous function of B_t, X_t is clearly $\mathcal{F}_t$-measureable.

We finally need to show that $\mathsf{E}(X_{t+\Delta t}|\mathcal{F}_t) = X_t$ for $\Delta t > 0$. Let us write $B_{t+\Delta t} = B_t + \Delta B$. Then

$$\begin{aligned} &\mathsf{E}(X_{t+\Delta t}|\mathcal{F}_t) \\ &= \mathsf{E}\left(e^{(t+\Delta t)/2} \sin(B_t + \Delta B) |\mathcal{F}_t\right) \\ &= e^{(t+\Delta t)/2} \mathsf{E}(\sin(B_t)\cos(\Delta B) + \cos(B_t)\sin(\Delta B) |\mathcal{F}_t) \\ &\overset{(i)}{=} e^{(t+\Delta t)/2} [\sin(B_t) \mathsf{E}(\cos(\Delta B) |\mathcal{F}_t) + \cos(B_t) \mathsf{E}(\sin(\Delta B) |\mathcal{F}_t)] \\ &\overset{(ii)}{=} e^{(t+\Delta t)/2} [\sin(B_t) \mathsf{E}(\cos(\Delta B)) + \cos(B_t) \mathsf{E}(\sin(\Delta B))] \\ &\overset{(iii)}{=} e^{(t+\Delta t)/2} \left[\sin(B_t) e^{-\Delta t/2} + 0\right] \\ &= e^{t/2} \sin(B_t) = X_t. \end{aligned} \tag{18}$$

Here (i) and (ii) are formulae for the conditional expectation. Moreover, for (iii), $\mathsf{E}(\cos(\Delta B))$ is listed and $\mathsf{E}(\sin(\Delta B))$ is obviously 0.

3 Problem

(a) Show that a linear stochastic differential equation

$$dX_t = p(t) X_t dt + q(t) dB_t \tag{19}$$

may be solved by an integrating factor $h(t)$ such that

$$d[h(t) X_t] = h(t) q(t) dB_t. \tag{20}$$

It follows from Itô's formula that

$$d(h(t) X_t) = h'(t) X_t dt + h(t) dX_t. \tag{21}$$

We then multiply Eqn. 19 by $h(t)$:

$$h(t) dX_t = d(h(t) X_t) - h'(t) X_t dt = h(t) p(t) X_t dt + h(t) q(t) dB_t. \tag{22}$$

The stated form follows if we find h such that

$$-h'(t) = h(t) p(t). \tag{23}$$

Then the solution to Eqn. 19 is:

$$h(t) X_t - h(t_0) X_{t_0} = \int_{t_0}^{t} h(t) q(t) dB_t, \tag{24}$$

or

$$X_t = \frac{h(t_0) X_{t_0} + \int_{t_0}^{t} h(t) q(t) dB_t}{h(t)}. \tag{25}$$

(b) Apply the method in (a) to solve the equation

$$dX_t = \frac{1}{t} X_t dt + t dB_t, \ t \geq 1, \ X_1 = 1. \tag{26}$$

The function h has to satisfy the equation

$$h' + \frac{1}{t} h = 0, \tag{27}$$

that is, $h(t) = t^{-1}$. In this case the start is at $t = 1$ such that

$$X_t = \frac{X_1 + \int_1^t \frac{1}{s} s dB_s}{1/t} = t(1 + B_t - B_1) = t\tilde{B}_t^{1,1}, \ t \geq 1. \tag{28}$$

4 Problem

Consider the following geometric Brownian motion in $\mathbb{R}^1$: $X_t = x \exp(-t + B_t), \ 0 \leq t$.

(a) Show that X_t is an Itô diffusion with generator

$$A = -\frac{x}{2}\frac{d}{dx} + \frac{x^2}{2}\frac{d^2}{dx^2}. \tag{29}$$

The diffusion representation of X_t is

$$dX_t = -X_t dt + X_t dB_t + \frac{1}{2} X_t dt$$
$$= -\frac{X_t}{2} dt + X_t dB_t. \tag{30}$$

The formula for the generator follows from

$$A = \beta \frac{d}{dx} + \frac{1}{2}\sigma^2 \frac{d^2}{dx^2}, \tag{31}$$

when $dX_t = \beta dt + \sigma dB_t$.

(b) We start X_t at $x = 1$ and define, for $x_0 < 1$,

$$\tau_0(\omega) = \min\{t \; ; \; X_t(\omega) \leq x_0\}. \tag{32}$$

Prove that

$$\mathsf{E}\tau_0 = -\log x_0. \tag{33}$$

We are going to apply Dynkin's Formula and need to know that $E\tau_0 < \infty$. Here the part E^{-t} will effectively "kill" the Brownian motion part E^{B_t}. Of course, X_t will also cross the level x_0 a.s. This argument is sufficient for the exam, but the full proof could be based on the law of the iterated logarithm, or the following simple estimate:

$$P(\tau_0 \geq u) \leq P(X_u \geq x_0)$$
$$= P(B_u \geq \log x_0 + u)$$
$$= P\left(\frac{B_u}{u^{1/2}} \geq \frac{\log x_0 + u}{u^{1/2}}\right) \tag{34}$$
$$= O\left(u^{-1/2} e^{-u}\right),$$

when $u \to \infty$; using the inequality in the list. This is sufficient for $E\tau_0$ to be finite.

The next step is to find an f such that Af is equal to a constant. Here $f(x) = \log x$ will do since

$$-\frac{x}{2}\frac{d\log x}{dx} + \frac{x^2}{2}\frac{d^2 \log x}{dx^2} = -1. \tag{35}$$

By Dynkin's Formula we then have

$$\mathsf{E}^1(f(X_\tau)) = \log(x_0) = \log 1 + \mathsf{E}^1\left(\int_0^{\tau_0} (-1)\, ds\right)$$
$$= -\mathsf{E}\tau_0, \tag{36}$$

or

$$\mathsf{E}\tau_0 = -\log(x_0). \tag{37}$$

(c) Let a, b be two positive numbers, $a < b$. We start X_t at $x \in [a, b]$ and let

$$p = \Pr(X_t \text{ hits level } b \text{ before it hits level } a)$$

Determine p for all $x \in [a, b]$.

We need to find a function $f(x)$ such that $Af = 0$, and try x^γ:

$$-\frac{x}{2}\frac{dx^\gamma}{dx} + \frac{x^2}{2}\frac{d^2x^\gamma}{dx^2} = x^\gamma\left(-\frac{\gamma}{2} + \frac{1}{2}\gamma(\gamma-1)\right) = 0, \tag{38}$$

for $\gamma = 2$. We then apply Dynkin's formula with the function $f(x) = x^\gamma$, $\gamma = 2$. The expected escape time from the interval $[a, b]$ is clearly finite since the situation will be similar to the case in (b) for the lower level a. Thus

$$\mathsf{E}(X_\tau^\gamma) = p \cdot b^\gamma + (1-p) \cdot a^\gamma = x^\gamma + 0, \tag{39}$$

and hence,

$$p = \frac{x^\gamma - a^\gamma}{b^\gamma - a^\gamma}. \tag{40}$$

A list of useful formulae

Note: The list does not state requirements for the formulae to be valid.

The probability density of a 1D Gaussian variable with mean μ and variance σ^2:

$$\phi(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{41}$$

A definite integral:

$$\int_{-\infty}^{\infty} \cos(x) \exp\left(-\frac{x^2}{2\sigma^2}\right) dx = \sqrt{2\pi}\sigma e^{-\sigma^2/2}. \tag{42}$$

An inequality:

Let X be $\mathcal{N}(0,1)$ and $x > 0$. Then

$$P(X \geq x) \leq \sqrt{\frac{1}{2\pi}} \frac{1}{x} e^{-x^2/2}. \tag{43}$$

Two formulae for Conditional Expectations:

$$\begin{array}{ll} (i) & \text{If } Y \text{ is } \mathcal{H}\text{-measurable, then } \mathsf{E}(XY|\mathcal{H}) = Y\mathsf{E}(X|\mathcal{H}). \\ (ii) & \text{If } X \text{ is independent of } \mathcal{H}\text{, then } \mathsf{E}(X|\mathcal{H}) = \mathsf{E}(X). \end{array}$$

The Itô isometry:

$$\mathsf{E}\left|\int_0^T f(t,\omega)\, dB_t(\omega)\right|^2 = \int_0^T \left(\mathsf{E}|f(t,\omega)|^2\right) dt = \|f\|^2_{L^2(\Omega\times[0,T])} \tag{44}$$

Itô's 2D formula:

$$dg(t, X_t, Y_t) = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial x}dX_t + \frac{\partial g}{\partial y}dY_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dX_t)^2 + \frac{\partial^2 g}{\partial x \partial y}dX_t dY_t + \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(dY_t)^2. \tag{45}$$

and "the rules".

The generator:

$$A(f)(x) = \sum_{i=1}^{n} \beta_i(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^{n} \left(\sigma(x)\sigma(x)'\right)_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(x). \tag{46}$$

Dynkin's Formula:

$$\mathsf{E}^x(f(X_\tau)) = f(x) + \mathsf{E}^x\left(\int_0^\tau Af(X_s)\, ds\right). \tag{47}$$