MA8109 Stokastiske metoder i systemteori
Autumn 2007
Suggested solution and extra comments
(Revised version December 10)

1 Problem

Consider the probability space $\{\Omega, \mathcal{F}, P\}$ and three sets A_1, A_2, $A_3 \in F$ where $A_1 \cup A_2 \cup A_3 = \Omega$. Moreover, the sets are disjoint ($A_i \cap A_j = \varnothing$ whenever $i \neq j$), and $P(A_i) > 0$ for $i = 1, 2, 3$.

(a) List the sets in the σ-algebra $\mathcal{H}$ generated by A_1, A_2, and A_3.

(b) The function Y from Ω into R is $\mathcal{H}$-measurable. Show that Y is equal to a constant on each of the sets A_1, A_2, and A_3. (Hint: Consider $\{\omega; X(\omega) = a\}$)

(c) Using the result from (b), compute the conditional expectation $E(X|\mathcal{H})(\omega)$ for an arbitrary stochastic variable X.

Solution:

(a) The definition of a σ-algebra is found in the notes. The σ-algebra $\mathcal{H}$ generated by A_1, A_2, and A_3 is the formally smallest σ-algebra containing A_1, A_2 and A_3. Using the definition, it is obvious that $\mathcal{H}$ contains A_1, A_2, A_3, Ω, $\varnothing$, and, in addition the 3 sets $A_1^c = A_2 \cup A_3$, $A_2^c = A_1 \cup A_3$, and $A_3^c = A_1 \cup A_2$.

(b) When Y is $\mathcal{H}$-measurable, then for all sets B in the Borel algebra of $\mathbb{R}$,

$$X^{-1}(B) = \{\omega; X(\omega) \in B\} \in \mathcal{H}. \tag{1}$$

Thus, if $X(\omega_1) = a$ for an $\omega_1 \in A_1$, then $\{\omega; X(\omega) = a\}$ will have to be equal to A_1, $A_1 \cup A_2$, $A_1 \cup A_3$, or Ω. In any case, X will have to be constant on A_1. The same argument works for A_2, and A_3.

(c) If we do not differ between functions equal a.s., the conditional expectation $\mathsf{E}(X|\mathcal{H})$ will be $\mathcal{H}$-measurable. From (b), we then know that it is constant on each of the sets A_1, A_2, and A_3, and it remains to determine the constants, say $a_i = X(\omega)$ for $\omega \in A_i$. Applying the definition, we have for all $H \in \mathcal{H}$,

$$\int_H \mathsf{E}(X|\mathcal{H})\, dP = \int_H X dP. \tag{2}$$

Then, using $H = A_1$, A_2, and A_3, we obtain

$$\int_{A_i} a_i dP = a_i P(A) = \int_{A_i} X dP, \tag{3}$$

or

$$a_i = \frac{\int_{A_i} X dP}{P(A)}, \; i = 1, 2, 3. \tag{4}$$

2 Problem

(a) Give a brief explanation of an adapted, elementary function, ϕ, and define the corresponding Itô integral,

$$I(\omega) = \int_S^T \phi(t, \omega)\, dB_t(\omega). \tag{5}$$

State the expectation and variance of I.

(b) Compute the expectation and the variance of the Itô integral

$$\int_0^1 (B_t^2 - t) dB_t \tag{6}$$

(Hint: If X is normal with mean μ and variance σ^2, then $E(X-\mu)^4 = 3\sigma^4$)

(c) Let F_t be the filtration w.r.t. 1D Brownian motion. Prove that

$$M_t = B_t^2 - t \tag{7}$$

is an F_t-Martingale.

Solution:

(a) An adapted function needs first of all a filtration. In the present case, this is the filtration $\mathcal{F}_t$ defined by the Brownian motion, that is, $\mathcal{F}_t$ is the σ-algebra generated by $\{B_s\}_{0\leq s\leq t}$. A function adapted to $\mathcal{F}_t$ is a random process, say $X(t)$, where $X(t)$ is $\mathcal{F}_t$-measurable for all t-s. An elementary function ϕ is a process that is constant on each set of a partition P of an interval $[S,T]$. The partition consists of all intervals $[t_k, t_{k+1}]$ defined by

$$S = t_0 < t_1 < \cdots < t_{n-1} < t_n = T \tag{8}$$

and

$$\phi(t,\omega) = \sum_{k=0}^{n-1} e_k(\omega) \chi_{[t_k,t_{k+1})}(t). \tag{9}$$

For ϕ to be $\mathcal{F}_t$-adapted, we need that $e_k(\omega)$ is $\mathcal{F}_{t_k}$-adapted for each k. In order to be in $\mathcal{V}[S,T]$, we also require that the variance stated below is finite, and a measurability condition (B.Ø. Def. 3.1.4). The Itô-integral of ϕ is defined as

$$I(\omega) = \int_S^T \phi(t,\omega)\, dB_t(\omega) = \sum_{k=0}^{n-1} e_k(\omega) \left(B_{t_{k+1}}(\omega) - B_{t_k}(\omega)\right). \tag{10}$$

Since $e_k(\omega)$ and $\Delta B_k = B_{t_{k+1}} - B_{t_k}$ are independent, $\mathsf{E} I = 0$. Moreover, using the Itô Isometry,

$$\begin{aligned} \operatorname{Var} I^2 = \mathsf{E} I^2 &= \mathsf{E}\left(\int_S^T |\phi(t,\omega)|^2\, dt\right) \\ &= \int_S^T \mathsf{E}\,|\phi(t,\omega)|^2\, dt = \sum_{k=0}^{n-1} \mathsf{E}\,|e_k|^2 (t_{k+1} - t_k). \end{aligned} \tag{11}$$

(This is also easy to see directly using the properties of e_k and ΔB_k).

(b) This is an Itô-integral since $(B_t^2 - t)$ is $\mathcal{F}_t$-adapted. Using the Itô Isometry, we first compute

$$\mathsf{E}(B_t^2 - t)^2 = \mathsf{E}\left(B_t^4 - 2tB_t^2 + t^2\right) = 3t^2 - 2t^2 + t^2 = 2t^2. \tag{12}$$

Then,

$$\operatorname{Var}\left(\int_0^1 (B_t^2 - t) dB_t\right) = \int_0^1 2t^2 = \frac{2}{3}. \tag{13}$$

(c) The shortest proof of this is to observe (using Itô's Formula) that

$$d\left(B_t^2 - t\right) = -dt + 2B_t dB_t + \frac{1}{2}2\left(dB_t\right)^2 = 2B_t dB_t. \tag{14}$$

Thus,

$$B_t^2 - t = 2\int_0^t B_s dB_s, \tag{15}$$

and all Itô-integrals are $\mathcal{F}_t$-martingales (B.Ø. Cor. 3.2.6).

Alternatively, checking the martingale definitions, $B_t^2 - t \ \subset L^2(\Omega) \subset L^1(\Omega)$, and also adapted to $\mathcal{F}_t$. Finally, for $0 \le s < t$, and $\Delta B = B_t - B_s$,

$$\begin{aligned}
\mathsf{E}\left(B_t^2 - t|\mathcal{F}_s\right) &= \mathsf{E}\left(\left(B_s + \Delta B\right)^2 - t|\mathcal{F}_s\right) \\
&= \mathsf{E}\left(B_s^2 + 2B_s\Delta B + \Delta B^2 - t|\mathcal{F}_s\right) \\
&= B_s^2 + 2B_s\mathsf{E}\Delta B + (t-s) - t \\
&= B_s^2 - s.
\end{aligned} \tag{16}$$

3 Problem

(a) Solve the 1D stochastic differential equation

$$dX_t = (1 - X_t)\,dt + dB_t,\ t \ge 0, \tag{17}$$

where $X_0 = Z$. Here Z has mean μ and variance σ^2 and is independent of the Brownian motion. Write down the time varying mean and the variance of the solution (Hint: Apply a suitable integrating factor).

(b) Assume that X_t and Y_t satisfy the stochastic differential equations ($X_t, Y_t, B_t \in \mathbb{R}$):

$$\begin{aligned}
dX_t &= \alpha X_t dt + Y_t dB_t,\ X_0 = x_0, \\
dY_t &= \alpha Y_t dt - X_t dB_t,\ Y_0 = y_0.
\end{aligned} \tag{18}$$

Derive and solve the differential equation for $R_t = X_t^2 + Y_t^2$.

Solution:

(a) The equation may be transformed into the class of linear equations we have considered by introducing $Y_t = X_t - 1$. The trick with an integrating factor may also be applied directly by multiplying the equation with a function $h(t)$ and observe that $h(t)\,dX_t = d(h(t)\,X_t) - X_t h'(t)\,dt$:

$$d(h(t)\,X_t) - X_t h'(t)\,dt = h(t)\,dt - X_t h(t)\,dt + h(t)\,dB_t.$$

For $h(t) = e^t$ we obtain

$$d\left(e^t X_t\right) = e^t dt + e^t dB_t,$$

or

$$X_t = Ze^{-t} + \left(1 - e^{-t}\right) + \int_0^t e^{s-t} dB_s.$$

Finally,

$$\mathsf{E}X_t = e^{-t}\mathsf{E}Z + \left(1-e^{-t}\right) + \mathsf{E}\int_0^t e^{s-t}dB_s = e^{-t}\mu + \left(1-e^{-t}\right),$$
$$\begin{aligned}\operatorname{Var} X_t &= e^{-2t}\sigma^2 + \operatorname{Var}\int_0^t e^{s-t}dB_s = e^{-2t}\sigma^2 + \int_0^t e^{2(s-t)}ds \\ &= e^{-2t}\sigma^2 + \frac{1}{2}\left(1-e^{-2t}\right).\end{aligned}$$

(Note that Z is also independent of $\int_0^t e^{s-t}dB_s$).

(b) In this case, the 2D process (X_t, Y_t) is transformed into the 1D process $R_t = X_t^2 + Y_t^2$. We need the multidimensional Itô Formula for $g(x,y) = x^2 + y^2$, which is this case, since $\partial^2 g/\partial x \partial y = 0$, will be

$$dg(x,y) = 2xdx + 2ydy + \frac{2}{2}(dx)^2 + \frac{2}{2}(dy)^2. \tag{19}$$

Thus, also introducing the rule $(dB_t)^2 = dt$,

$$\begin{aligned} dR_t &= 2X_t\left(\alpha X_t dt + Y_t dB_t\right) + 2Y_t\left(\alpha Y_t dt - X_t dB_t\right) + Y_t^2 dt + X_t^2 dt \\ &= (2\alpha+1)\left(X_t^2 + Y_t^2\right)dt = (2\alpha+1)R_t dt \end{aligned} \tag{20}$$

The solution of this (ordinary) diff. equation follows immediately

$$R_t = \left(x_0^2 + y_0^2\right)e^{(2\alpha+1)t}. \tag{21}$$

4 Problem

(a) Define the generator A of an autonome Itô diffusion

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \; X_t \in \mathbb{R}^n, B_t \in \mathbb{R}^m. \tag{22}$$

Express the solution $u(t,x)$, $t>0$, $x \in R^n$, of the problem

$$\begin{aligned}\frac{\partial u}{\partial t} &= Au, \\ u(0,x) &= f(x), \end{aligned} \tag{23}$$

in terms of f and X_t. Show how this gives an explicit formula for the solution when the diffusion is ordinary Brownian motion.

The Ornstein–Uhlenbeck process is an Itô diffusion and a simple 1D model for physical Brownian motion. Consider the special case

$$dX_t = -X_t dt + dB_t. \tag{24}$$

Let $0 < c < C$ and consider the stopping time

$$\tau_{c,C} = \inf\left\{t \geq 0; \; X_0 = c, \; X_t = 0 \text{ or } X_t = C\right\}. \tag{25}$$

It is known that $E(\tau_{c,C}) < \infty$.

(Hint for (b) and (c): The differential equation $-xy' + \frac{1}{2}y'' = 1$ *has the general solution*

$$y(x) = C_1 + C_2 g(x) + y_p(x), \tag{26}$$

where

$$\begin{aligned} g(x) &= \int_0^x e^{s^2} ds, \\ y_p(x) &= \sqrt{\pi} \int_0^x \operatorname{erf}(s)\, e^{s^2} ds, \\ \operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds. \end{aligned} \tag{27}$$

(b) Compute the probability that X_t *hits the level* C *before it hits* 0.

(c) Express $E(\tau_{c,C})$ *in terms of the functions in Eqn. 27. Determine* $E(\tau_c)$*, where* $\tau_c = \inf\{t \geq 0;\ X_0 = c,\ X_t = 0\}$.

Solution

(a) The generator is the differential operator

$$A = b(x) \cdot \nabla + \frac{1}{2} \sum_{i,j=1}^{n} \left(\sigma\sigma^t\right)_{ij} \frac{\partial^2}{\partial x_i \partial y_j}. \tag{28}$$

The solution may be expressed as

$$u(t,x) = \mathsf{E}^x f(X_t). \tag{29}$$

The probability density for a Brownian motion at time t starting at 0 for $t = 0$ is

$$\varphi(x) = \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{2t}\right). \tag{30}$$

Thus, is that particular case,

$$u(t,x) = \mathsf{E}^x f(B_t) = \int_y f(y)\, \varphi(x-y)\, d^n y. \tag{31}$$

(b) This is an application of Dynkin's formula: If $\mathsf{E}\tau < \infty$, then for an $f \in C_c^2(\mathbb{R}^n)$,

$$\mathsf{E}^x f(X_\tau) = f(x) + \mathsf{E}^x \int_0^\tau A f(X_s)\, ds. \tag{32}$$

In the present case, the generator is the operator

$$-x\frac{d}{dx} + \frac{1}{2}\frac{d^2}{dx^2}, \tag{33}$$

and the idea is to find a nice $f \in C_c^2(\mathbb{R})$ such that $Af = 0$ on the interval $[0, C]$ Here, the general solution of the equation

$$-x\frac{df}{dx} + \frac{1}{2}\frac{d^2 f}{dx^2} = 0 \tag{34}$$

is given, and $g(x)$ will work if we modify it with a smooth transition to 0 outside $[0, C]$, e.g. $f(x) = g(x)\theta(x)$ where $\theta \in C_c^2(\mathbb{R})$, $\theta(x) = 1$ on $[0, C]$. Let p_C be the probability we are looking for. Then from Dynkin's Lemma,

$$p_C g(C) + (1 - p_C) g(0) = g(c) + 0. \tag{35}$$

Since $g(0) = 0$,we obtain

$$p_C = \frac{g(c)}{g(C)}. \tag{36}$$

(c) We still use Dynkin's Formula, and need that $Af = 1$ on the interval $[0, C]$. Therefore, y_p (actually $y_p(x)\theta(x)$) is feasible since $Ay_p(X_t) = 1$ as long as $X_t \in [0, C]$. Then,

$$p_C y_p(C) + (1 - p_C) y_p(0) = y_p(c) + \mathsf{E}^x(\tau_{c,C}). \tag{37}$$

Hence,

$$\begin{aligned} \mathsf{E}^x(\tau_{c,C}) &= p_C y_p(C) - y_p(c) \\ &= \frac{g(c)}{g(C)} y_p(C) - y_p(c) \\ &= \frac{y_p(C)}{g(C)} g(c) - y_p(c). \end{aligned} \tag{38}$$

If we look at the definitions of g and y_p, we observe first of all that

$$\frac{y_p(C)}{g(C)} = \frac{\sqrt{\pi}\int_0^C \operatorname{erf}(s) e^{s^2} ds}{\int_0^C e^{s^2} ds} < \sqrt{\pi}. \tag{39}$$

and, moreover,

$$\lim_{C \to \infty} \frac{\int_0^C \operatorname{erf}(s) e^{s^2} ds}{\int_0^C e^{s^2} ds} = 1. \tag{40}$$

Thus, $\mathsf{E}^x(\tau_{c,C})$ is uniformly bounded:

$$\mathsf{E}^c(\tau_{c,C}) < \sqrt{\pi} g(c) - y_p(c) = \sqrt{\pi}\int_0^c (1 - \operatorname{erf}(s)) e^{s^2} ds. \tag{41}$$

Since we also have

$$\tau_{c,C}(\omega) \underset{C \to \infty}{\nearrow} \tau_c(\omega) \tag{42}$$

for all paths, we obtain by the Monotone Convergence Theorem that $\mathsf{E}^c(\tau_{c,C}) \underset{C \to \infty}{\nearrow} \mathsf{E}^c(\tau_c)$, or

$$\mathsf{E}^c(\tau_c) = \sqrt{\pi}\int_0^c (1 - \operatorname{erf}(s)) e^{s^2} ds. \tag{43}$$

Digression: Observe that it is essential that Af in Dynkin's formula does not cause problems for us when $|x| \to \infty$. Even if $Ay_p(x) = 1$ for all values of x, we can *not* write something like

$$y_p(0) = y_p(c) + \mathsf{E}^c \int_0^{\tau_c} Ay_p(X_s) ds = y_p(c) + \mathsf{E}^c(\tau_c),$$

which leads to the absurd result

$$\mathsf{E}^c(\tau_c) = -y_p(c)!$$

We need to ensure that we are able to taper off f by a function like θ above, and that is impossible if we just consider the interval $[0, \infty)$.