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MA8109 Stochastic Processes in Systems Theory
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Exercise set 1

1 Let X, Y, I be sets, and $f : X \to Y$ a function. Prove the following statements:

a) If $A, B \subset X$, then $(A \cap B)^c = A^c \cup B^c$.

b) If $A_i \subset X$ for $i \in I$, then $(\bigcup_{i\in I} A_i)^c = \bigcap_{i\in I} A_i^c$.

c) If $A \subset X$, then $f^{-1}(A^c) = f^{-1}(A)^c$.

d) If $A_i \subset X$ for $i \in I$, then $f^{-1}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f^{-1}(A_i)$.

Hint: $x \in \bigcup_{i\in I} A_i$ iff $\exists j \in I$ such that $x \in A_j$, and $x \in \bigcap_{i\in I} A_i$ iff $\forall i \in I$, $x \in A_i$.

2 Find the σ-algebra on a set X generated by:

a) $A, B \subset X$ where $A \cap B = \emptyset$.

b) $\mathcal{A} = \{\{i\}\}_{i\in\mathbb{N}}$ where $X = \mathbb{N}$.

3 (Øksendal, Problem 2.3)
Prove that for any collection $\{\mathcal{H}_i\}_{i\in I}$ of σ-algebras,

$$\mathcal{H} = \cap_{i\in I}\mathcal{H}_i$$

is a σ-algebra as well.

Hint: Verify all properties a σ-algebra should fulfill.

4 Let $(X, \mathcal{F}, m)$ be a measure space and $A_i \in \mathcal{F}$ for $i \in \mathbb{N}$. Prove that

a) If $A_1 \subset A_2$, then $m(A_1) \le m(A_2)$.

b) $m(\bigcup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} m(A_i)$.

c) If $A_1 \subset A_2 \subset A_3 \subset \ldots$ and $A = \bigcup_{i=1}^{\infty} A_i$, then $m(A) = \lim_{i\to\infty} m(A_i)$.

Hint for b) and c): Make disjoint unions, use σ-additivity of m.

5 Let m_L be the Lebesgue measure on $\mathbb{R}$. Prove that $m_L(\mathbb{Q}) = 0$.

Hint: Let $\mathbb{Q} = \bigcup_{i=1}^{\infty}$ and consider the intervals $(q_i - 2^{-i}\epsilon, q_i + 2^{-i}\epsilon)$.

6 Let $f_1, f_2 : X \to \mathbb{R}$ be $\mathcal{F}$-measurable. Prove that $f(x) = \max(f_1(x), f_2(x))$ is $\mathcal{F}$-measurable.

Hint: Enough to prove that $f^{-1}((a, \infty)) \in \mathcal{F}$ for all $a \in \mathbb{R}$.

7 Let $(X, \mathcal{F}, m)$ be a measure space and $f_1, f_2 : A \to [0, \infty)$ $\mathcal{F}$-measurable. Prove that

$$\int (f_1 + f_2) dm = \int f_1 dm + \int f_2 dm.$$

Hint: Prove it for simple functions, approximate general functions, and go to the limit using MCT.

8 Let $(X, \mathcal{F}, m)$ be a measure space and $\phi : X \to [0, \infty)$ be $\mathcal{F}$-measurable. Prove that

$$\mu(\emptyset) = 0, \qquad \mu(A) = \int_A \phi \, dm \qquad \text{for any} \qquad A \in \mathcal{F},$$

defines a measure on $(X, \mathcal{F})$.

9 Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \to \mathbb{R}$ a random variable on it. Define

$$\mu_X(A) = m(X^{-1}(A)) \qquad \text{for any} \qquad A \in \mathcal{F}.$$

a) Show that μ_X is a probability measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.

Hint: $X^{-1}(\cap A_i) = \cap X^{-1}(A_i)$, $X^{-1}(\cup A_i) = \cup X^{-1}(A_i)$.

b) Let $f : \mathbb{R} \to [0, \infty)$ be $\mathcal{B}_{\mathbb{R}}$-measurable. Show that

$$E(f(X) = \int_X f(X)) \, dP(\omega) = \int_{\mathbb{R}} f(x) \, d\mu_X(x).$$

Hint: Prove it for simple functions, then approximate f, and go to the limit. You may use that $s_n \nearrow f$ then also $s_n \circ X \nearrow f \circ X$.

10 The Monotone and Dominated Convergence theorems both establish situations where the limits below are equal:

$$(1) \qquad \lim_{n\to\infty} \int_\Omega f_n(\omega) \, dP(\omega) \quad \text{and} \quad \int_\Omega \left(\lim_{n\to\infty} f_n(\omega) \right) dP(\omega).$$

Let $\Omega = [0, 1]$ and P the Lebesgue measure. Consider the following sequence of functions $\{f_n\}$ defined on Ω:

$$(2) \qquad f_n(\omega) = \begin{cases} a_n \frac{\omega}{\omega_n}, & 0 \le \omega \le \omega_n, \\ a_n \left(2 - \frac{\omega}{\omega_n}\right), & \omega_n \le \omega \le 2\omega_n, \\ 0, & 2\omega_n \le \omega \le 1, \end{cases}$$

where $\lim_{n\to\infty} \omega_n = 0$ (make a sketch!).

(a) Prove that $g(\omega) = \lim_{n\to\infty} f_n(\omega) = 0$ for all values of $\omega \in [0, 1]$.

(b) Compute $\int g dP$ and $\lim_{n\to\infty} \int f_n dP$ when $a_n = \omega_n^{-1/2}, \omega_n^{-1}$, and ω_n^{-2}.

(c) For which cases in (b) will the function $h(\omega) = \max_n f_n(\omega)$ not be integrable?

11 a) Prove that if $X \leq Y$, then $E(X|\mathcal{H}) \leq E(Y|\mathcal{H})$.

Hint: Use that $E(X|\mathcal{H}) \geq 0$ if $X \geq 0$.

b) Show that if $0 \leq X_1 \leq X_2 \leq \cdots \leq X_n \underset{n\to\infty}{\longrightarrow} X$, then $E(X_n|\mathcal{H}) \underset{n\to\infty}{\longrightarrow} E(X|\mathcal{H})$ a.e. (Note that $X_n(\omega) \underset{n\to\infty}{\longrightarrow} X(\omega)$ for *all* $\omega \in \Omega$ if we accept $+\infty$ as a limit for positive functions).

Hint: Use the Monotone Convergence Theorem and a) to prove that the limit function $Y = \lim_{n\to\infty} E(X_n|\mathcal{H})$ exists and satisfies all conditions of $E(X|\mathcal{H})$. Also use that a pointwise limit of a sequence of $\mathcal{H}$-measurable functions is $\mathcal{H}$-measurable (This is a general result from measure theory. A sequence $\{f_n\}$ converges *pointwise* to f if $f_n(\omega) \underset{n\to\infty}{\longrightarrow} f(\omega)$ for all ω).

12 a) Prove that if the covariance matrix $\mathbf{\Sigma}$ is non-singular (and hence positive definite!), then

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\mathbf{u}'\mathbf{x}}\, e^{i\mathbf{u}'\mu-\frac{1}{2}\mathbf{u}'\mathbf{\Sigma}\mathbf{u}}\, d^n u = \frac{1}{(2\pi)^{n/2}\left|\mathbf{\Sigma}\right|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\mu)'\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)}. \tag{3}$$

Hint: Introduce new variables $\mathbf{y} = \mathbf{\Sigma}^{1/2}\mathbf{u}$, so that the integral splits into a product of one-dimensional integrals. Use that for $a \in \mathbb{R}$,

$$\int_{\mathbb{R}} \exp\left(iya - \frac{1}{2}y^2\right) dy = \sqrt{2\pi}e^{-\frac{a^2}{2}}.$$

b) Assume that $\mathbf{X}$ is a multivariate Gaussian variable and $\mathsf{E}(\mathbf{X}) = 0$. Show, by taking appropriate partial derivatives of the characteristic function that:

(i) "*The expectation of triple products always vanishes*":

$$\mathsf{E}(X_1X_2X_3) = 0. \tag{4}$$

(ii) "*The Fourth-Cumulant Identity*":

$$\mathsf{E}(X_1X_2X_3X_4) = \mathsf{E}(X_1X_2)\,\mathsf{E}(X_3X_4) + \mathsf{E}(X_1X_3)\,\mathsf{E}(X_2X_4) + \mathsf{E}(X_1X_4)\,\mathsf{E}(X_2X_3). \tag{5}$$

Hint: Use the Taylor expansion to find the derivative of the characteristic function,

$$\phi(\mathbf{u}) = \exp\left(-\frac{1}{2}\mathbf{u}'\mathbf{\Sigma}\mathbf{u}\right) = 1 - \frac{1}{2}\mathbf{u}'\mathbf{\Sigma}\mathbf{u} + \frac{1}{2}\left(\frac{1}{2}\mathbf{u}'\mathbf{\Sigma}\mathbf{u}\right)^2 + \cdots.$$

Look up *Wikipedia* or *MathWorld* for an explanation of the term *cumulant.*