# MA8109 Stochastic Processes in Systems Theory Autumn 2013

**Exercise set 2**

**1** (Øksendal 2:2.15) Prove that Brownian motion has the following fractal behavior: If $B_t$ is a standard one-dimensional Brownian motion and $c > 0$, then the process

$$Y_t = \frac{1}{c} B_{c^2 t} \tag{1}$$

is also a standard Brownian motion.

**2** (Øksendal 2:2.12 and 2:2.15) Let $B_t$ be Brownian motion on $\mathbb{R}^d$, $t_0 > 0$, and $U \in \mathbb{R}^{d\times d}$ such that $U^T U = I$.

Show that the following processes also are Brownian motions on $\mathbb{R}^d$:

**a)** $\tilde{B}_t = B_{t+t_0} - B_{t_0}$.

**b)** $\tilde{B}_t = U B_t$.

**3** (Øksendal 2:2.13) Let $B_t$ be a Brownian motion in $\mathbb{R}^2$ and $\rho > 0$. Compute

$$P(|B_t| < \rho).$$

**Hint:** $B_t$ is $N(\vec{0}, tI)$, use the probability density function.

**4** (Øksendal 2:2.8) Let $B_t$ be a Brownian motion in $\mathbb{R}$.

**a)** Show that $E(e^{iuB_t}) = e^{-\frac{1}{2}u^2 t}$ and use this to prove that $E(B_t^4) = 3t^2$.

**b)** Use $E(f(B_t)) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(x) e^{-\frac{x^2}{2t}} dx$ and integration by parts to show that $E(B_t^4) = 3t^2$.

**5** Consider the standard Brownian motion in $\mathbb{R}^n$, $B_t$. Prove that

$$\mathsf{E}\left(|B_t - B_s|^4\right) = n\,(n+2)\,(t-s)^2\,. \tag{2}$$

**Hint**: For this problem you could use that

$$\mathsf{E}\left(X_1^2 + \cdots + X_n^2\right)^2 = \mathsf{E}\left(\sum_{i,j=1}^{N} X_i^2 X_j^2\right), \tag{3}$$

and use the independence and distribution of the increments and the fact that $E(B_i^4(t)) = 3t$ for $i = 1, \ldots n$ (previous exercise).

**6** Let $B_t$ be a Brownian motion on $\mathbb{R}$, and let $0 < t_1 < t_2 < \cdots < t_n$.

**a)** Show that the characteristic function of $B_{t_1}$ and $B_{t_2}$

$$\Phi_{B_{t_1},B_{t_1}}(u_1, u_2) = e^{\frac{1}{2}u^T\Sigma u},$$

for $u = (u_1, u_2)^T$ and $\Sigma = \begin{pmatrix} t_1 & t_1 \\ t_1 & t_2 \end{pmatrix}$.

**Hint:** Write $B_{t_2} = B_{t_1} + (B_{t_2} - B_{t_1})$ and use independence of the R.V.'s on the right. What can you say about characteristic functions of independet R.V.'s?

**b)** Show that $B_t$ is a Gaussian process.

**Hint:** Show that $B_{t_1}, \ldots, B_{t_n}$ is multivariate Gaussian by considering the characteristic function. Obs: You need not compute the convariance matrix explicitly to conclude.

**7** Let $B_t$ be a Brownian motion and $\mathcal{F}_t$ the filtration generated by $B_t$. Show that $B_t^2 - t$ is a martingale w.r.t. $\mathcal{F}_t$.

**Hint:** $B_t^p = B_s^p + (B_t^p - B_s^p)$ for $p = 1, 2$, use different properties of the conditional expectation.

**8** Let $B_t$ be a Brownian motion in $\mathbb{R}^1$ and $\mathcal{F}_t$ the filtration generated by $B_t$.

**a)** Show that $E((e^{B_t})^p) = e^{\frac{1}{2}p^2 t}$ for $p \in \mathbb{N}$. ($e^{B_t}$ is geometric B.M.)

**Hint:** Integrate using the probability density function, complete the squares.

**b)** Show that $e^{B_t - \frac{1}{2}t}$ is a martingale w.r.t. $\mathcal{F}_t$.

**Hint:** $B_t = B_s + (B_t - B_s)$, use a) and different properties of the conditional expectation.

**9** Let $B_t$ be a Brownian motion on $\mathbb{R}$ with filtration $\{\mathcal{F}_t\}_t$,

$$0 = t_0 < t_1 < t_2 < \ldots,$$

$\Delta B_{t_i} = B_{t_i} - B_{t_{i-1}}$ and $h(\omega, t)$ an $\{\mathcal{F}_t\}_t$-adapted stochastic process such that $E(h^2(\cdot, t)) < \infty$ for each $t$. Verify that the following process is a $\{\mathcal{F}_{t_i}\}_i$-martingale (i.e. a discrete martingale):

$$Y_0 = 0, \qquad Y_n = \sum_{i=1}^{n} h(\omega, t_{i-1})\Delta B_{t_i} \quad \text{for} \quad n \in \mathbb{N}.$$

**10** Let $B_t$ be a Brownian motion in $\mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_n = 1$, and $\tilde{B}_t$ the linear interpolant of $\{(t_i, B_{t_i})\}_i$: For every $k = 0, \ldots, n$,

$$\tilde{B}_t = B_{t_k} + \frac{t - t_k}{t_{k+1} - t_k}(B_{t_{k+1}} - B_{t_k}) \quad \text{for} \quad t \in [t_k, t_{k+1}].$$

Show that:

**a)** $\tilde{B}_0 = 0$ a.s. and $E(\tilde{B}_t) = 0$.

**b)** $|E(\tilde{B}_t\tilde{B}_s) - \min(s, t)| \leq \max_k |t_{k+1} - t_k|$.

**c)** $\tilde{B}_t$ is a Gaussian process.

**Hint c):** Use characteristic functions. Similar computation as in Exercise 6.

11 Let $X_1, X_2, \dots$ be i.i.d. $N(0,1)$ random variables. Define

$$\Delta B_1^0 = X_1,$$
$$\Delta B_{2m-1}^n = \Delta B_m^{n-1} + \frac{1}{2^{\frac{n+1}{n}}} X_{i(n,m)},$$
$$\Delta B_{2m}^n = \Delta B_m^{n-1} - \frac{1}{2^{\frac{n+1}{n}}} X_{i(n,m)},$$

for $n = 1, 2, 3, \dots$, $m = 1, \dots, 2^{n-1}$ and $i(n,m) = 2^{n-1} + m$.

Prove by induction that for each fixed $n$,

$$\Delta B_1^n, \dots, \Delta B_{2^n}^n$$

are independent $N(0, \frac{1}{2^n})$ random variables.

**Hint:** You may assume that $\Delta B_1^n, \dots, \Delta B_{2^n}^n$ are multivariate Gaussian, so zero correlation shows independence.

[If $X \in \mathbb{R}^d$ is multivariate Gaussian, then $AX$ is also so for any $A \in \mathbb{R}^{d \times d}$]