MA8109
Stochastic Processes in Systems Theory
Autumn 2013

Exercise set 2 – Solutions

1 This is straightforward: The process Y_t is clearly Gaussian, the increments will be independent ($t \to c^2 t$ is increasing in t), and

$$\mathrm{Var}\,(Y_t) = \frac{1}{c^2}\,\mathrm{Var}\,(B_{c^2 t}) = \frac{1}{c^2} c^2 t = t.$$

2 **a)** We check the axioms of Brownian motion for $\tilde{B}_t = B_{t+t_0} - B_{t_0}$.
$\tilde{B}_0 = B_{t_0} - B_{t_0} = 0$ a.s., and since $B_t - B_s \sim N(\mathbf{0}, (t-s)I)$, it follows that

$$\tilde{B}_t - \tilde{B}_s = B_{t+t_0} - B_{s+t_0} \sim N(\mathbf{0}, (t + t_0 - s - t_0)I) = N(\mathbf{0}, (t-s)I).$$

Finally, let $0 < t_1 < \ldots < t_n$. Then

$$\begin{aligned} \tilde{B}_{t_1} &= B_{t_1+t_0} - B_{t_0}, \\ \tilde{B}_{t_2} - \tilde{B}_{t_1} &= B_{t_2+t_0} - B_{t_1+t_0}, \\ &\vdots \\ \tilde{B}_{t_n} - \tilde{B}_{t_{n-1}} &= B_{t_n+t_0} - B_{t_{n-1}+t_0}. \end{aligned}$$

are all independent since this property holds for the $B_{t_k+t_0} - B_{t_{k-1}+t_0}$'s.

b) We have $\tilde{B}_t = UB_t$ and we check the axioms. For $t = 0$, $\tilde{B}_0 = UB_0 = \mathbf{0}$ a.s. The increments, $\tilde{B}_t - \tilde{B}_s = U(B_t - B_s)$, are linear combinations of Gaussian variables, so they are also Gaussian. We check that

$$\mathrm{E}(\tilde{B}_t - \tilde{B}_s) = \mathrm{E}(U(B_t - B_s)) = U\mathrm{E}(B_t - B_s) = U \cdot \mathbf{0} = \mathbf{0},$$

and

$$\begin{aligned} \mathrm{Var}(\tilde{B}_t - \tilde{B}_s) &= \mathrm{Var}(U(B_t - B_s)) = \mathrm{E}\left[(U(B_t - B_s))\,(U(B_t - B_s))^T\right] \\ &= \mathrm{E}\left[U(B_t - B_s)(B_t - B_s)^T U^T\right] = U\mathrm{Var}\,(B_t - B_s)\,U^T \\ &= U(t-s)IU^T = (t-s)I. \end{aligned}$$

Therefore

$$\tilde{B}_t - \tilde{B}_s \sim N(\mathbf{0}, (t-s)I), \text{ for } t > s \geq 0.$$

As for the increments we have,

$$\begin{aligned} \tilde{B}_{t_1} &= UB_{t_1}, \\ \tilde{B}_{t_2} - \tilde{B}_{t_1} &= U\,(B_{t_2} - B_{t_1}), \\ &\vdots \\ \tilde{B}_{t_n} - \tilde{B}_{t_{n-1}} &= U\left(B_{t_n} - B_{t_{n-1}}\right). \end{aligned}$$

These $\tilde{B}$-increments are multivariate Gaussian distributed since the B-increments are. We the show independence by considering the covariance,

$$\begin{aligned} &\operatorname{Cov}\left(\tilde{B}_{t_i} - \tilde{B}_{t_{i-1}}, \tilde{B}_{t_j} - \tilde{B}_{t_{j-1}}\right) \\ &= \mathrm{E}\left(U(B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})^T U\right) \\ &= U\mathbf{0}U^T = \mathbf{0} \qquad (1) \end{aligned}$$

for $i \neq j$, where we have used that the increments in the original Brownian motion B_t are independent and their covariances are zero.

3 B_t is a Brownian motion, meaning $B_t \sim N(\mathbf{0}, tI)$. Letting $\mathbf{x} = (x, y) \in \mathbb{R}^2$, this means that

$$P(|B_t| < \rho) = \iint\limits_{|\mathbf{x}|<\rho} \frac{e^{-\frac{\mathbf{x}^T I \mathbf{x}}{2t}}}{2\pi t} \mathrm{d}\mathbf{x} = \iint\limits_{|\mathbf{x}|<\rho} \frac{e^{-\frac{x^2+y^2}{2t}}}{2\pi t} \mathrm{d}\mathbf{x}.$$

By changing to polar coordinates, we now obtain

$$P(|B_t| < \rho) = \int_0^{\rho}\int_0^{2\pi} \frac{e^{-\frac{r^2}{2t}}}{2\pi t} r \mathrm{d}\theta \mathrm{d}r = 1 - e^{-\frac{\rho^2}{2t}}.$$

4 **a)** Since B_t is a Brownian Motion, then by definition we have $B_t \sim N(0, t)$. In addition, we know that

$$\int_{-\infty}^{\infty} e^{-\frac{(x-a)^2}{b}} dx = \sqrt{\pi b}$$

where $b > 0$. Therefore,

$$\begin{aligned} E(e^{iuB_t}) &= \int_{\mathbb{R}} e^{iux} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \\ &= \frac{e^{-\frac{u^2 t}{2}}}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-iut)^2}{2t}} dx \\ &= \frac{e^{-\frac{u^2 t}{2}}}{\sqrt{2\pi t}} \cdot \sqrt{2\pi t} = e^{-\frac{1}{2}u^2 t}. \end{aligned}$$

Differentiating we find that

$$\begin{aligned} \frac{d}{du} E(e^{iuB_t}) &= \frac{d}{du} e^{-\frac{1}{2}u^2 t} = -ut E(e^{iuB_t}) \\ \frac{d^2}{du^2} E(e^{iuB_t}) &= (u^2 t^2 - t) E(e^{iuB_t}) \\ \frac{d^3}{du^3} E(e^{iuB_t}) &= (-u^3 t^3 + 3ut^2) E(e^{iuB_t}) \\ \frac{d^4}{du^4} E(e^{iuB_t}) &= (u^4 t^4 - 6u^2 t^3 + 3t^2) E(e^{iuB_t}). \end{aligned}$$

Differentiating under the integral sign (the integrand is smooth with bounded derivatives, use the dominated convergence theorem), we also see that

$$\frac{d^k}{du^k} E(e^{iuB_t}) \,|_{u=0} = E\left(\frac{\partial^k}{\partial u^k} e^{iuB_t}\Big|_{u=0}\right) = i^k E\left(B_t^k\right).$$

Hence

$$E(B_t^4) = i^4 \frac{d^4}{du^4} E(e^{iuB_t})\big|_{u=0} = 3t^2.$$

b) Obviously by the condition,

$$\begin{aligned}
E(B_t^4) &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} x^4 e^{-\frac{x^2}{2t}} dx \\
&= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} \left(\frac{x^4 t}{-x}\right) de^{-\frac{x^2}{2t}} \\
&= \frac{1}{\sqrt{2\pi t}} \left[(-x^3 t e^{-\frac{x^2}{2t}})\big|_{-\infty}^{\infty} - \int_{\mathbb{R}} (-3x^2 t e^{-\frac{x^2}{2t}}) dx \right] \\
&= \frac{3}{\sqrt{2\pi t}} \int_{\mathbb{R}} x^2 t e^{-\frac{x^2}{2t}} dx = \frac{3}{\sqrt{2\pi t}} \int_{\mathbb{R}} (-x t^2) de^{-\frac{x^2}{2t}} \\
&= \frac{3}{\sqrt{2\pi t}} \left[(-x t^2 e^{-\frac{x^2}{2t}})\big|_{-\infty}^{\infty} - \int_{\mathbb{R}} (-t^2 e^{-\frac{x^2}{2t}}) dx \right] \\
&= \frac{3t^2}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{x^2}{2t}} dx = \frac{3t^2}{\sqrt{2\pi t}} \cdot \sqrt{2\pi t} = 3t^2.
\end{aligned}$$

5 We assume that $B_0 = 0$. Set $X = B_t - B_s = (X_1, \cdots, X_n)$, so that X is multivatiate Gaussian with independent components, $\mathsf{E} X_i = 0$, and $\operatorname{Var} X_i = t - s$. Moreover,

$$\mathsf{E}\, |X|^4 = \mathsf{E}\left(X_1^2 + \cdots + X_n^2\right)^2 = \mathsf{E}\left(\sum_{i,j=1}^{N} X_i^2 X_j^2\right).$$

If $i = j$, then, by the Fourth Cumulant Identity (or a previous exercise), $\mathsf{E}\left(X_i^4\right) = 3\left(\mathsf{E} X_i^2\right)^2 = 3\,(t-s)^2$. There are n such terms. When $i \neq j$, $\mathsf{E}\left(X_i^2 X_j^2\right) = \mathsf{E}\left(X_i^2\right) \mathsf{E}\left(X_j^2\right) = (t-s)^2$, and we have $n\,(n-1)$ of these. Altogether,

$$\mathsf{E}\, |X|^4 = 3\,(t-s)^2\, n + n\,(n-1)\,(t-s)^2 = n\,(n+2)\,(t-s)^2.$$

6 **a)** Consider $B_{t_1}, B_{t_2}, \ldots, B_{t_n}$ for $0 < t_1 < t_2 < \ldots < t_n$.
The *characteristic function* is defined as

$$\Phi_{B_{t_1}, B_{t_2}, \ldots, B_{t_n}}(u_1, u_2, \ldots, u_n) := E\left[\exp\left\{ i \sum_{j=1}^{n} u_j B_{t_j} \right\}\right].$$

Since

$$B_{t_j} = \sum_{k=1}^{j} \Delta B_k \qquad \text{for} \qquad \Delta B_k = (B_{t_k} - B_{t_{k-1}}) \quad \text{and} \quad B_{t_0} := B_0 = 0,$$

we introduce $\gamma_k = \sum_{j=k}^{n} u_j$ and write

$$\Phi_{B_{t_1}, B_{t_2}, \ldots, B_{t_n}}(u_1, u_2, \ldots, u_n) = E\left[\exp\left\{ i \sum_{k=1}^{n} \gamma_k \Delta B_k \right\}\right]$$

Using the independence of the ΔB_k's,

$$\Phi_{B_{t_1},B_{t_2},\ldots,B_{t_n}}(u_1,u_2,\ldots,u_n) = E\left[\exp\left\{i\sum_{k=1}^{n}\gamma_k\Delta B_k\right\}\right]$$

$$= E\left[\prod_{k=1}^{n}\exp\left\{i\gamma_k\Delta B_k\right\}\right] = \prod_{k=1}^{n}E\left[\exp\left\{i\gamma_k\Delta B_k\right\}\right] = \prod_{k=1}^{n}\Phi_{\Delta B_k}(\gamma_k).$$

Since $\Delta B_k \sim N(0,\sigma_k^2)$ for $\sigma_k^2 = t_k - t_{k-1}$, we know from earlier that

$$\Phi_{\Delta B_k}(\gamma_k) = \exp\left\{-\frac{\gamma_k^2\sigma_k^2}{2}\right\},$$

and hence

$$\begin{aligned}
&\Phi_{B_{t_1},B_{t_2},\ldots,B_{t_n}}(u_1,u_2,\ldots,u_n)\\
&= E\left[\exp\left\{-\frac{1}{2}\gamma^\intercal\Lambda\gamma\right\}\right]\\
&= E\left[\exp\left\{-\frac{1}{2}\mathbf{u}^\intercal P^\intercal\Lambda P\mathbf{u}\right\}\right]\\
&= E\left[\exp\left\{-\frac{1}{2}\mathbf{u}^\intercal\Sigma\mathbf{u}\right\}\right]
\end{aligned}$$

where $\Lambda = diag(\sigma_1^2,\sigma_2^2,\ldots,\sigma_n^2)$,

$$P = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1\\ 0 & 1 & 1 & \cdots & 1 & 1\\ 0 & 0 & 1 & \cdots & 1 & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & 1\\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

and, finally

$$\begin{aligned}
\Sigma &= \begin{bmatrix} \sigma_1^2 & \sigma_1^2 & \sigma_1^2 & \cdots & \sigma_1^2 & \sigma_1^2\\ \sigma_1^2 & \sigma_1^2+\sigma_2^2 & \sigma_1^2+\sigma_2^2 & \cdots & \sigma_1^2+\sigma_2^2 & \sigma_1^2+\sigma_2^2\\ \sigma_1^2 & \sigma_1^2+\sigma_2^2 & \sigma_1^2+\sigma_2^2+\sigma_3^2 & \cdots & \sigma_1^2+\sigma_2^2+\sigma_3^2 & \sigma_1^2+\sigma_2^2+\sigma_3^2\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \sigma_1^2 & \sigma_1^2+\sigma_2^2 & \sigma_1^2+\sigma_2^2+\sigma_3^2 & \cdots & \sum_{k=1}^{n-1}\sigma_k^2 & \sum_{k=1}^{n}\sigma_k^2 \end{bmatrix}\\
&= \begin{bmatrix} t_1 & t_1 & t_1 & \cdots & t_1 & t_1\\ t_1 & t_2 & t_2 & \cdots & t_2 & t_2\\ t_1 & t_2 & t_3 & \cdots & t_3 & t_3\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ t_1 & t_2 & t_3 & \cdots & t_{n-1} & t_{n-1}\\ t_1 & t_2 & t_3 & \cdots & t_{n-1} & t_n \end{bmatrix}
\end{aligned}$$

One can see that this answers the question by taking $n = 2$

b) A random vector $\mathbf{X}$ multivariate Gaussian $N_n(\mathbf{x};\mu,\Sigma)$ if and only if its characteristic function is

$$\Phi_{\mathbf{X}}(\mathbf{u}) = \exp\left\{i\mathbf{u}^\intercal\mu - \frac{1}{2}\mathbf{u}^\intercal\Sigma\mathbf{u}\right\},$$

for some $\mu \in \mathbb{R}^n$ and $\Sigma \in \mathbb{R}^{n\times n}$.

By part (a), this is the case for $B_{t_1}, \ldots, B_{t_n}$ (with $\mu = 0$) for any n and $0 < t_1 < \cdots < t_n$, and hence B_t is a Gaussian process.

7 Since B_t is $\mathcal{F}_t$-measurable, so is $B_t^2 - t$,

$$E[|B_t^2 - t|] \leq E[|B_t - B_0|^2] + E[|t|] = E[(B_t - B_0)^2] + E[t] = 2t < \infty,$$

and for $t \leq s$,

$$\begin{aligned} E[B_s^2 - s|\mathcal{F}_t] &= E[B_s^2|\mathcal{F}_t] - s \\ &= E[(B_t + B_s - B_t)^2|\mathcal{F}_t] - s \\ &= E[B_t^2|\mathcal{F}_t] + 2E[B_t(B_s - B_t)|\mathcal{F}_t] + E[(B_s - B_t)^2|\mathcal{F}_t] - s \\ &= B_t^2 + 2B_tE[B_s - B_t] + E[(B_s - B_t)^2] - s \\ &= B_t^2 + 0 + s - t - s \\ &= B_t^2 - t. \end{aligned}$$

Here we used that B_t^2 and B_t are $\mathcal{F}_t$-measurable, and that $B_s - B_t$ and $(B_s - B_t)^2$ are independent of $\mathcal{F}_t$. It follows that $B_t^2 - t$ is a Martingale.

8 **a)** Note that $B_t \sim \mathcal{N}(0, t)$ and calculate

$$\begin{aligned} E(e^{pB_t}) &= \int_{\mathbb{R}} e^{px}\ dF_{B_t}(x) \\ &= \int_{-\infty}^{\infty} e^{px} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\ dx \\ &= e^{\frac{1}{2}p^2 t}, \end{aligned}$$

for all $p \in \mathbb{C}$.

b) Let $Y_t = e^{B_t - t/2}$. To see that Y_t is a $\mathcal{F}_t$-martingale, we note that it is $\mathcal{F}_t$-measurable since it is a continuous function of a $\mathcal{F}_t$-measureable fuction. By a), we also have

$$E(|Y_t|) = E(e^{B_t})e^{-t/2} < \infty.$$

Finally, for $t \leq s$,

$$\begin{aligned} E(Y_s|\mathcal{F}_t) &= E(e^{B_s}e^{-s/2}|\mathcal{F}_t) \\ &= E(e^{B_s}e^{-s/2}|\mathcal{F}_t) \\ &= E(e^{B_s - B_t}e^{-s/2}e^{B_t}|\mathcal{F}_t) \\ &= e^{B_t}E(e^{(B_s - B_t)}e^{-s/2}|\mathcal{F}_t) \\ &= e^{B_t}E(e^{(B_s - B_t)}e^{-s/2}) \\ &= e^{B_t}e^{(s-t)/2}e^{-s/2} \\ &= e^{B_t}e^{-t/2} \\ &= Y_t \end{aligned}$$

where we used properties of the conditional expectation with e^{B_t} $\mathcal{F}_t$-measurable and $e^{B_s - B_t}$ independent of $\mathcal{F}_t$ (a Borel function of an independent R.V. is independent).

9 First we check the process Y_i is $\mathcal{F}_{t_i}$-measurable, for all $i = 0, \cdots, n$. This is obvious since $h(\omega, t_i)$ and ΔB_{t_i} are $\mathcal{F}_{t_i}$-adapted for all $t \in [0, T]$, and Y_n is constructed as

$$Y_0 = 0, \text{ and } Y_n = \sum_{i=1}^{n} h(\omega, t_{i-1}) \Delta B_{t_i}.$$

Next, by Cauchy-Schwartz and independence of ΔB_{t_j} and $\mathcal{F}_{t_{j-1}}$,

$$\begin{aligned} E(|Y_n|) &= E\left(|\sum_{i=1}^{n} h(\cdot, t_{i-1}) \Delta B_{t_i}| \right) \\ &\leq \sum_{i=1}^{n} E\left(|h(\cdot, t_{i-1}) \Delta B_{t_i}| \right) \\ &\leq \sum_{i=1}^{n} \sqrt{E(|h(\cdot, t_{i-1})|^2)} \sqrt{E(|\Delta B_{t_i}|^2)} \\ &= \sum_{i=1}^{n} \sqrt{E(|h(\cdot, t_{i-1})|^2)(t_i - t_{i-1})} \\ &< +\infty, \end{aligned}$$

because we assumed $E\left(h^2(\cdot, t)\right) < +\infty$ for each t.

Finally, the conditional expectation

$$\begin{aligned} E(Y_n | \mathcal{F}_{t_j}) &= E\left(\sum_{i=1}^{n} h(\cdot, t_{i-1}) \Delta B_{t_i} \,\middle|\, \mathcal{F}_{t_j} \right) \\ &= \sum_{i=1}^{j} E\left(h(\cdot, t_{i-1}) \Delta B_{t_i} \,\middle|\, \mathcal{F}_{t_j} \right) + \sum_{i=j+1}^{n} E\left(h(\cdot, t_{i-1}) \Delta B_{t_j} \,\middle|\, \mathcal{F}_{t_j} \right) \\ &= \sum_{i=1}^{j} h(\omega, t_{i-1}) \Delta B_{t_i} + \sum_{i=j+1}^{n} E\left(E\left(h(\cdot, t_{i-1}) \Delta B_{t_i} \,\middle|\, \mathcal{F}_{t_{j-1}} \right) \,\middle|\, \mathcal{F}_{t_j} \right) \\ &= \sum_{i=1}^{j} h(\omega, t_{i-1}) \Delta B_{t_i} + \sum_{i=j+1}^{n} E\left(h(\cdot, t_{i-1}) E(\Delta B_{t_i}) \,\middle|\, \mathcal{F}_{t_j} \right) \\ &= Y_j + 0, \end{aligned}$$

since $E(\Delta B_{t_i}) = 0$. Hence Y_n is a martingale with respect to $\{\mathcal{F}_{t_i}\}_i$.

10 **a)** This is almost immediate, as $\tilde{B}_0 = B_{t_0} = 0$ a.s and $E(\tilde{B}_t) = 0$ by linearity.

b) Suppose without loss of generality that $s \in [t_k, t_{k+1}], t \in [t_l, t_{l+1}]$ with $s \leq t, k \leq l$. Then

$$\begin{aligned} \tilde{B}_t \tilde{B}_s = &B_{t_l} B_{t_k} + \frac{s - t_k}{t_{k+1} - t_k} B_{t_l}(B_{t_{k+1}} - B_{t_k}) + \frac{t - t_l}{t_{l+1} - t_l}(B_{t_{l+1}} - B_{t_l}) B_{t_k} \\ &+ \frac{(t - t_l)(s - t_k)}{(t_{l+1} - t_l)(t_{k+1} - t_k)}(B_{t_{l+1}} - B_{t_l})(B_{t_{k+1}} - B_{t_k}). \end{aligned}$$

Now by assumption on the order of the points we have

$$\begin{aligned}
E(B_{t_l}B_{t_k}) &= t_k\\
E(B_{t_l}(B_{t_{k+1}}-B_{t_k})) &= \min(t_l,t_{k+1})-t_k = (t_{k+1}-t_k)(1-\delta_{kl})\\
E((B_{t_{l+1}}-B_{t_l})B_{t_k}) &= 0\\
E((B_{t_{l+1}}-B_{t_l})(B_{t_{k+1}}-B_{t_k})) &= (t_{k+1}-t_k)\delta_{kl},
\end{aligned}$$

whence

$$\begin{aligned}
|E(\tilde{B}_t\tilde{B}_s)-s| &= |-(s-t_k)+(s-t_k)(1-\delta_{kl})+\frac{(t-t_k)(s-t_k)}{t_{k+1}-t_k}\delta_{kl}|\\
&= (s-t_k)\frac{t_{k+1}-t}{t_{k+1}-t_k}\delta_{kl} \le s-t_k \le t_{k+1}-t_k\\
&\le \max_{0\le k\le n-1}(t_{k+1}-t_k).
\end{aligned}$$

c) Suppose that $0=\tau_0\le\tau_1<\tau_2<\cdots<\tau_m\le 1$ (the τ_0 is only for simpler notation) and that $\tau_j\in[t_{k(j)},t_{k(j)+1}]$ for each j (note that $k:\{0,\dots,m\}\to\{0,\dots,n-1\}$ need not be injective). Now (where $u_0=0$)

$$\begin{aligned}
\sum_{j=1}^m u_j\tilde{B}_{\tau_j} &= \sum_{j=1}^m u_j\left(B_{t_{k(j)}}+\frac{\tau_j-t_{k(j)}}{t_{k(j)+1)}-t_{k(j)}}(B_{t_{k(j)+1}}-B_{t_{k(j)}})\right)\\
&= \sum_{j=0}^m\left(\frac{\tau_j-t_{k(j)}}{t_{k(j)+1}-t_{k(j)}}u_j+\sum_{i=j+1}^m u_i\right)(B_{t_{k(j)+1}}-B_{t_{k(j)}})\\
&= \sum_{l=0}^{n-1}\left[\sum_{j\in k^{-1}(l)}\left(\frac{\tau_j-t_l}{t_{l+1}-t_l}u_j+\sum_{i=j+1}^m u_i\right)\right](B_{t_{l+1}}-B_{t_l}),
\end{aligned}$$

and hence, using the independence and distribution of $\Delta B_{t_{l+1}}=B_{t_{l+1}}-B_{t_l}$,

$$\begin{aligned}
&\Phi_{\tilde{B}_{\tau_1},\dots,\tilde{B}_{\tau_m}}(u_1,\dots,u_m) = E\left(\exp\left(i\sum_{j=1}^m u_j\tilde{B}_{\tau_j}\right)\right)\\
&= \prod_{l=0}^{n-1}\Phi_{\Delta B_{t_{l+1}}}\left(\sum_{j\in k^{-1}(l)}\left(\frac{\tau_j-t_l}{t_{l+1}-t_l}u_j+\sum_{i=j+1}^m u_i\right)\right)\\
&= \exp\left(-\frac{1}{2}\sum_{l=0}^{n-1}\left[\sum_{j\in k^{-1}(l)}\left(\frac{\tau_j-t_l}{t_{l+1}-t_l}u_j+\sum_{i=j+1}^m u_i\right)\right]^2(t_{l+1}-t_l)\right).
\end{aligned}$$

Since the exponent is clearly a quadratic form in $u_1,\dots,u_m$ with non-positive coefficients, we have that $(\tilde{B}_{\tau_1},\dots,\tilde{B}_{\tau_m})\sim N(0,\Sigma)$. The covariance matrix Σ is determined by the coefficients of u_iu_j.

11 The conclusion obviously holds when $n=0$. We proceed by induction: fix some $n\ge 1$, and assume that the conclusion holds for $n-1$. Observe that each component of $B=(\Delta B_1^n,\Delta B_2^n,\dots,\Delta B_{2^n}^n)$ is a linear combination of the independent $N(0,1)$ variables $X_1,X_2,\dots,X_{2^n}$. This immediately shows that B is multivariate Gaussian, with zero mean. Using that normal variables are independent if and only if they are uncorrelated, what remains is to show that the covariance matrix is $\frac{1}{2^n}I$.

To compute the variances of the components of B, note that

$$\begin{aligned}
&E\left[(\Delta B^n_{2m-1})^2\right] \\
&= E\left[\left(\frac{1}{2}\Delta B^{n-1}_m + \frac{1}{2^{\frac{n+1}{2}}}X_{i(n,m)}\right)^2\right] \\
&= \frac{1}{4}E\left[(\Delta B^{n-1}_m)^2\right] + 2E\left[\left(\frac{1}{2}\Delta B^{n-1}_m\right)\left(\frac{1}{2^{\frac{n+1}{2}}}X_{i(n,m)}\right)\right] + \frac{1}{2^{n+1}}E\left[(X_{i(n,m)})^2\right] \\
&= \frac{1}{4}\frac{1}{2^{n-1}} + 0 + \frac{1}{2^{n+1}} \\
&= \frac{1}{2^n},
\end{aligned}$$

where we used both the induction hypothesis, together with the fact that ΔB^{n-1}_m and $X_{i(n,m)}$ are uncorrelated since ΔB^{n-1}_m is a linear combination of $X_1, ..., X_{2^{n-1}}$ and $i(n,m) > 2^{n-1}$. A similar computation shows that $E\left[(\Delta B^n_{2m})^2\right] = \frac{1}{2^n}$.
To show that the components of B are uncorrelated, note that

$$\begin{aligned}
E\left[(\Delta B^n_{2m-1})(\Delta B^n_{2m})\right] &= E\left[\left(\frac{1}{2}\Delta B^{n-1}_m + \frac{1}{2^{\frac{n+1}{2}}}X_{i(n,m)}\right)\left(\frac{1}{2}\Delta B^{n-1}_m - \frac{1}{2^{\frac{n+1}{2}}}X_{i(n,m)}\right)\right] \\
&= \frac{1}{4}E\left[(\Delta B^{n-1}_m)^2\right] - \frac{1}{2^{n+1}}E\left[(X_{i(n,m)})^2\right] \\
&= 0.
\end{aligned}$$

For the other pairs of components we get a computation looking like

$$E\left[(\Delta B + X)(\Delta\tilde{B} + \tilde{X})\right] = 0,$$

where each pair of $\Delta B,\ X,\ \Delta\tilde{B},\ \tilde{X}$ is uncorrelated. This concludes the proof.