Norwegian University of Science and Technology
Department of Mathematical Sciences

MA8109 Stochastic Processes in Systems Theory
Autumn 2013

Exercise set 3

1 Consider the Itô-integral

$$I(\omega) = \int_0^1 f(t,\omega)\, dB_t(\omega)$$

where the function $f(t,\omega)$ is pre-determined (deterministic), that is,

$$f(t,\omega) = g(t) \quad \text{for all } \omega \in \Omega.$$

a) Show that f is F_t-adapted and that $f \in \mathcal{V}(0,1)$ if and only if $\int_0^1 |g(t)|^2\, dt < \infty$.

b) Determine the probability distribution of I.
Hint: Apply App. A, Thm. A 19 in Øksendal.

2 Prove from the definition of the Itô integral, that

$$\int_0^t s\, dB_s = tB_t - \int_0^t B_s ds.$$

Hint: Note that $B_0 = 0$ a.s. Prove and apply Abel's summation by part formula:

$$\sum_{j=0}^{n-1} \Delta(a_j b_j) = \sum_{j=0}^{n-1} a_j \Delta b_j + \sum_{j=0}^{n-1} b_{j+1} \Delta a_j\ , \quad \Delta x_j = x_{j+1} - x_j.$$

Also note the alternate form, similar to the integration by parts formula:

$$\sum_{j=0}^{n-1} a_j \Delta b_j = a_n b_n - a_0 b_0 - \sum_{j=0}^{n-1} b_{j+1} \Delta a_j$$

3 (Øksendal 3:3.4) Check whether the following processes X_t are martingales w.r.t. $\{\mathcal{F}_t\}_t$.

(i) $X_t = B_t + 4t$

(ii) $X_t = B_t^2$

(iii) $X_t = t^2 B_t - 2\int_0^t s B_s ds$

(iv) $X_t = B_1(t)B_2(t)$ where (B_1, B_2) is 2-dimensional Brownian Motion.

4 Øksendal Exercise 4:4.1

5 Øksendal Exercise 4:4.4

6 Øksendal Exercise 4:4.7

7 Øksendal Exercise 4:4.8 a)

8 Øksendal Exercise 4:4.11

9 Øksendal Exercise 4:4.13
Hint: You may assume that $X_t, M_t \in L^p(\Omega)$ for all $p \in [1, \infty)$.

10 Øksendal Exercise 5:5.1 (i) and (iv)
Hint: Check *all* the axioms of the definition...

11 Øksendal Exercise 5:5.3

More hints at the end of Øksendal.