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MA8109 Stochastic Processes in Systems Theory Autumn 2013

Exercise set 3 – solutions

1 **a)** For each t, the stochastic variable $f(t,\omega)$ has a constant value. Hence f is F_t-adapted and $E(f(t,\cdot)) = g(t)$. All constants are measurable, and independent of all other stochastic variables. Moreover,

$$\|f\|^2_{L^2([0\ 1]\times\Omega)} = \int_0^1 \mathsf{E}\,|f(t,\omega)|^2\,dt = \int_0^1 |g(t)|^2\,dt,$$

which is finite (required for $f \in \mathcal{V}(0,1)$!) if and only if $\int_0^1 |g(t)|^2\,dt < \infty$.

b) Theorem A19 deals with the limit in $L^2(\Omega)$ of gaussian stochastic variables. The Itô integral for functions in $\mathcal{V}(S,t)$ is the $L^2(\Omega)$-limit of integrals of elementary functions. These integrals are finite sums of gaussian variables and themselves gaussian. The limit, $\int f(t)\,dB_t$, is therefore gaussian (with mean 0). By the Itô isometry,

$$\operatorname{Var}\left(\int_0^1 f(t)\,dB_t\right) = \left\|\int_0^1 f(t)\,dB_t\right\|^2_{L^2(\Omega)} = \|f\|^2_{L^2([0\ 1]\times\Omega)} = \int_0^1 |g(t)|^2\,dt.$$

Hence, $\int_0^1 f(t)\,dB_t$ is $\mathcal{N}\left(0, \int_0^1 |g(t)|^2\,dt\right)$.

2 We consider a partition $\mathcal{P}$ of the interval $[0,t]$ such that

$$0 = t_0 < t_1 < \cdots < t_n = t.$$

Then, as easily seen,

$$\int_0^t s dB_s = \lim_{\mathcal{P}\to 0} \sum_0^{n-1} t_j \left(B_{t_{j+1}} - B_{t_j}\right).$$

(Show that $\phi_n(s) = \sum_0^{n-1} t_j \chi_{[t_j,t_{j+1})}(s)$ converges to the function $f(s) = s$ in $L^2([0,t]\times\Omega)$!). Thus,

$$\begin{aligned}
\sum_0^{n-1} t_j \left(B_{t_{j+1}} - B_{t_j}\right) &= \sum_0^{n-1} t_j \Delta B_j \\
&= \sum_0^{n-1} \Delta(t_j B_j) - \sum_{j=0}^{n-1} B_{t_{j+1}} \Delta t_j \\
&= tB_t - 0B_0 - \sum_{j=0}^{n-1} B_{t_{j+1}} \Delta t_j.
\end{aligned}$$

Since the last sum is an approximation to the regular integral $\int_0^t B_s ds$, where the integrand is continuous with probability 1, the result follows immediately

3 **(i)** We check the main martingale property:

$$\mathsf{E}(X_s|\mathcal{F}_t) = \mathsf{E}(B_s + 4s|\mathcal{F}_t) = B_t + 4s \neq X_t.$$

(ii) The hint in BØ, Ed. 6, is just

$$\mathsf{E}(X_t) = \mathsf{E}\left(B_t^2\right) = nt + \mathsf{E}\left(B_0^2\right) \neq X_0,$$

and hence it violates that $\mathsf{E}(X_t)$ is constant (BØ assumes $B_t \in \mathbb{R}^n$).

It is also possible to check the main martingale property directly by writing $B_s = B_t + \Delta B$, that is, $\Delta B = B_s - B_t$ where $s > t$:

$$\begin{aligned}
\mathsf{E}\left(B_s^2|\mathcal{F}_t\right) &= \mathsf{E}\left((B_t + \Delta B)^2|\mathcal{F}_t\right) \\
&= \mathsf{E}\left(\left(B_t^2 + 2B_t\Delta B + (\Delta B)^2\right)|\mathcal{F}_t\right) \\
&= \mathsf{E}\left(B_t^2|\mathcal{F}_t\right) + 2B_t\mathsf{E}(\Delta B|\mathcal{F}_t) + \mathsf{E}\left((\Delta B)^2|\mathcal{F}_t\right) \\
&= B_t^2 + 0 + \mathsf{E}(\Delta B)^2 \\
&= B_t^2 + (s-t) \neq X_t.
\end{aligned}$$

(Generalize the argument to $\mathbb{R}^n$ yourself):

(iii) In this case, we also need to consider the *first two martingale properties* for $\{X_t, \mathcal{F}_t\}$: The integral is a.s. a regular Riemann integral, and all Riemann sums will be $\mathcal{F}_t$-measurable. The limit, which is a limit also in $L^2(\Omega)$, is then $\mathcal{F}_t$-measurable. The rest is $\mathcal{F}_t$-measurable and hence X_t is $\mathcal{F}_t$-measurable.

For the second property, it is easy to see that X_t is the sum of two functions in $L^2(\Omega)$ (recall that the integral may be written as a limit in $L^2(\Omega)$ of integrals of elementary functions). Thus, since $L^2(\Omega) \subset L^1(\Omega)$, we have that $X_t \in L^1(\Omega)$.

The last property is the most interesting (A short version of the argument is also found in BØ, Ed. 6). Assume that $t < s$:

$$\begin{aligned}
\mathsf{E}(X_s|\mathcal{F}_t) &= s^2\mathsf{E}(B_s|\mathcal{F}_t) - 2\mathsf{E}\left(\int_0^s uB_u du|\mathcal{F}_t\right) \\
&= s^2B_t - 2\mathsf{E}\left(\int_0^t uB_u du|\mathcal{F}_t\right) - 2\mathsf{E}\left(\int_t^s uB_u du|\mathcal{F}_t\right) \\
&= s^2B_t - 2\int_0^t uB_u du - 2\mathsf{E}\left(\int_t^s u(B_t + B_u - B_t)\,du|\mathcal{F}_t\right) \\
&= s^2B_t - 2\int_0^t uB_u du - 2B_t\int_t^s u du - 0 \\
&= s^2B_t - 2\int_0^t uB_u du - B_t\left(s^2 - t^2\right) \\
&= t^2B_t - 2\int_0^t uB_u du = X_t!
\end{aligned}$$

(iv) It is obvious that X_t is $\mathcal{F}_t$-measurable. Moreover, since B_1 is independent of

B_2,

$$\begin{aligned}\|B_{1t}B_{2t}\|_{L^1(\Omega)} &= \int_\Omega |B_{1t}(\omega)\, B_{2t}(\omega)|\, dP(\omega) \\ &= \int_\Omega |B_{1t}(\omega)|\, dP(\omega) \int_\Omega |B_{2t}(\omega)|\, dP(\omega) \\ &= \|B_{1t}\|_{L^1(\Omega)} \|B_{2t}\|_{L^1(\Omega)} < \infty.\end{aligned}$$

May also argue using Schwarz' inequality:

$$\begin{aligned}\|B_{1t}B_{2t}\|_{L^1(\Omega)} &= \int_\Omega |B_{1t}(\omega)\, B_{2t}(\omega)|\, dP(\omega) \\ &\le \|B_{1t}\|_{L^2(\Omega)} \|B_{2t}\|_{L^2(\Omega)} \ < \infty).\end{aligned}$$

Making the usual decomposition, $B_{is} = B_{it} + \Delta B_i$, we have

$$\begin{aligned}X_s &= \mathsf{E}(B_{1s}B_{2s}|\mathcal{F}_t) = \mathsf{E}((B_{1t}+\Delta B_1)(B_{2t}+\Delta B_2)|\mathcal{F}_t) \\ &= \mathsf{E}(B_{1t}B_{2t}|\mathcal{F}_t) + \mathsf{E}(\Delta B_1 B_{2t}|\mathcal{F}_t) + \mathsf{E}(\Delta B_2 B_{1t}|\mathcal{F}_t) + \mathsf{E}(\Delta B_1 \Delta B_2|\mathcal{F}_t) \\ &= \mathsf{E}(B_{1t}B_{2t}|\mathcal{F}_t) + B_{2t}\mathsf{E}(\Delta B_1|\mathcal{F}_t) + B_{1t}\mathsf{E}(\Delta B_2|\mathcal{F}_t) + 0 \\ &= B_{1t}B_{2t} + 0 = X_t!\end{aligned}$$

4 Recall *Itô's Formula* for the transformation $Y_t = g(t, X_t)$:

$$dY_t = g_t(t, X_t)\, dt + g_x(t, X_t)\, dX_t + \frac{1}{2} g_{xx}(t, X_t) \cdot (dX_t)^2$$

and "The Rules". E.g. when $dX_t = udt + vdB_t$, then $(dX_t)^2 = v^2 dt$.

a) Here $dX_t = dB_t$ and $g(t,x) = x^2$. Thus,

$$\begin{aligned}dY_t &= 2X_t dX_t + \frac{1}{2} 2 \cdot (dX_t)^2 \\ &= dt + 2B_t dB_t.\end{aligned}$$

b) Again, $dX_t = dB_t$, with $g(t,x) = 2 + t + e^x$:

$$\begin{aligned}dY_t &= 1dt + e^{X_t} dX_t + \frac{e^{X_t}}{2} \cdot (dX_t)^2 \\ &= \left(1 + \frac{e^{B_t}}{2}\right) dt + e^{B_t} dB_t.\end{aligned}$$

c) In this case, $X_t = (B_1(t), B_2(t))$ is 2D BM and $g(x_1, x_2) = x_1^2 + x_2^2$:

$$\begin{aligned}dY_t) &= 2B_1(t)\, dB_1(t) + 2B_2(t)\, dB_2(t) + \frac{1}{2}(2dt + 2dt) \\ &= 2dt + 2(B_1(t)\, dB_1(t) + B_2(t)\, dB_2(t)).\end{aligned}$$

d) This looks a little confusing until we write

$$g(t,x) = \begin{bmatrix} t_0 + t \\ x \end{bmatrix}$$

and obtain from Itô's formula

$$d\mathbf{X}_t = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x} dX_t = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} dB_t = \begin{bmatrix} dt \\ dB_t \end{bmatrix}.$$

e) We see straight away that

$$dX_1(t) = dB_1(t) + dB_2(t) + dB_3(t).$$

For $X_2(t)$ we get that

$$\begin{aligned} dX_2(t) &= d(B_2^2(t) - B_1(t)B_3(t)) \\ &= d(B_2^2(t)) - d(B_1(t)B_3(t)) \\ &= \big(2B_2(t)dB_2(t) + dt\big) - \big(B_3(t)dB_1(t) + B_1(t)dB_1(t)\big). \end{aligned}$$

In matrix form,

$$dX_t = \begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt + \begin{bmatrix} 1 & 1 & 1 \\ -B_3(t) & 2B_2(t) & -B_1(t) \end{bmatrix} \begin{bmatrix} dB_1(t) \\ dB_2(t) \\ dB_3(t) \end{bmatrix},$$

where we have used the fact that $d(B_t^2) = 2B_t dB_t + dt$ and that B_1, B_2, B_3 are independent Brownian motions.

5 **a)** Let

$$X_t = \int_0^t \theta(s,\omega)dB(s) - \frac{1}{2}\int_0^t |\theta(s,\omega)|^2 ds,$$

then $Z_t = e^{X_t} = g(t, X_t)$ and $dX_t = \theta(t,\omega)dB_t - \frac{1}{2}|\theta(t,\omega)|^2 dt$.
Since $g(t,x)$ is a C^2 and X_t is an Ito process ($\theta \in \mathcal{V}^n$), we use Ito's formula to deduce that

$$\begin{aligned} dZ_t &= \frac{\partial g}{\partial t}(t,X_t)dt + \frac{\partial g}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)(dX_t)^2 \\ &= 0 + Z_t(\theta dB_t - \frac{1}{2}|\theta|^2 dt) + \frac{1}{2}Z_t(\theta dB_t - \frac{1}{2}|\theta|^2 dt)^2 \\ &= Z_t\theta dB_t - \frac{1}{2}Z_t|\theta|^2 dt + \frac{1}{2}Z_t(\theta dB_t)^2 - \frac{1}{2}Z_t|\theta|^2\theta dB_t dt + \frac{1}{8}Z_t|\theta|^4(dt)^2 \\ &= Z_t\theta(t,\omega)dB(t). \end{aligned}$$

Here we used the rules $dB_i(t)dB_j(t) = dt\delta_{ij}$, $dt^2 = 0$, and $dB_t dt = 0$.

b) Since $Z_t\theta_k(t,\omega) \in \mathcal{V}[0,T]$, then

$$Z_t\theta(t,\omega) = (Z_t\theta_1(t,\omega), Z_t\theta_2(t,\omega) \cdots, Z_t\theta_n(t,\omega)) \in \mathcal{V}^n[0,T],$$

and by Corollary 3.2.6 in Øksendal,

$$Z_t = Z_0 + \int_0^t (Z_s\theta(s,\omega))\, dB_s$$

is a martingale for $t \le T$.

6 **a)** The simplest example is $v = (\delta_{1i})_i$ with $X_0 = 0$, yielding $X_t = B_{t,1}$. Then $E(X_t^2) = t \ne 0 = E(X_0^2)$ for $t > 0$, and so X_t^2 cannot be a martingale with respect to $\mathcal{F}_t^{(n)}$.

b) We define

$$M_t = |X_t|^2 - \int_0^t \|v_s\|^2 ds,$$

and want to show that this is a martingale with respect to the filtration $\mathcal{F}_t^{(n)}$. By Ito's lemma (with $g(t,x) = |x|^2$) we have

$$\begin{aligned} d|X_t|^2 &= 2X_t dX_t + \frac{1}{2} 2(dX_t)^2 \\ &= 2X_t v_t dB_t + \|v_t\|^2 dt, \end{aligned}$$

and hence

$$\begin{aligned} dM_t &= d|X_t|^2 - \|v_t\|^2 dt \\ &= 2X_t v_t dB_t. \end{aligned}$$

By Lemma 3.2.6 it is thus sufficient to show that $Xv \in \mathcal{V}^n(0,T)$. We check each point in the definition:

(i) Note that $X(\cdot,\omega)$ is continuous a.s and $X(t,\cdot)$ is $\mathcal{F}$ - measurable for each $t \in [0,T]$. Hence $X(t,\omega) = \lim_{n\to\infty} X(\lfloor tn \rfloor/n, \omega)$ for a.e $(t,\omega) \in [0,T] \times \Omega$, where each $(t,\omega) \mapsto X(\lfloor tn \rfloor/n,\omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, being piecewise constant in time. Thus X is $\mathcal{B} \times \mathcal{F}$-measurable. Since $v \in \mathcal{V}^n(0,T)$ each component of v is $\mathcal{B} \times \mathcal{F}$-measurable. Hence each component of Xv is $\mathcal{B} \times \mathcal{F}$-measurable.

(ii)' By the extension of Theorem 3.2.1 part (iv) we have that X_t is $\mathcal{F}_t^{(n)}$-adapted. Each component of v is $\mathcal{F}_t^{(n)}$-adapted since $v \in \mathcal{V}^n(0,T)$. Hence each component of Xv is $\mathcal{F}_t^{(n)}$-adapted.

(iii) If $v \in L^\infty([0,T] \times \Omega, \mathbb{R}^n)$ and $X_0 \in L^2(\Omega)$ then[1]

$$\begin{aligned} E\left[\int_0^T |X_t|^2 \|v_t\|^2 dt\right] &\leq \|v\|_\infty^2 E\left[\int_0^T |X_t|^2 dt\right] \\ &= \|v\|_\infty^2 \int_0^T E(|X_t|^2) dt \\ &\leq 2\|v\|_\infty^2 \int_0^T \left(E(|X_0|^2) + E\left[\left(\int_0^t v_s dB_s\right)^2\right]\right) dt \\ &= 2\|v\|_\infty^2 \int_0^T \left(E(|X_0|^2) + E\left[\int_0^t \|v_s\|^2 ds\right]\right) dt \\ &\leq 2T\|v\|_\infty^2 \left(E(|X_0|^2) + E\left[\int_0^T \|v_t\|^2 dt\right]\right) < \infty. \end{aligned}$$

Hence indeed $Xv \in \mathcal{V}^n(0,T)$, and M_t is a martingale with respect to the filtration $\mathcal{F}_t^{(n)}$.

7 Apply Ito's formula (Theorem 4.2.1 in Øksendal (p. 49)) to $Y_t = f(X_t)$ where $X_t = B_t$ ($u = 0$, $v = I$). Since $\frac{\partial f}{\partial t} = 0$ and $f(X_t) \in \mathbb{R}^1$, it follows that

$$\mathrm{d}Y_s = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x)\mathrm{d}(X_i)_s + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(x)\mathrm{d}(X_i)_s \mathrm{d}(X_j)_s.$$

[1] The Ito isometry extends to $\mathbb{R}^n$, but I cannot see a proof in Øksendal. It is not really necessary, since we can get a slightly weaker bound by using the isometry in $\mathbb{R}$ and then using Cauchy-Schwarz.

In our case the above simplifies to

$$\text{(1)} \qquad \mathrm{d}f(B_s) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)\mathrm{d}(B_i)_s + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}(x)\mathrm{d}s$$

since $\mathrm{d}(X_i)_s\mathrm{d}(X_j)_s = \mathrm{d}(B_i)_s\mathrm{d}(B_j)_s = \delta_{ij}\mathrm{d}s$ (where δ_{ij} is Kronecker's delta). By definition this means that

$$\begin{aligned} f(B_t) - f(B_0) &= \int_0^t \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)\mathrm{d}(B_i)_s + \frac{1}{2}\int_0^t \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}(x)\mathrm{d}s \\ &= \int_0^t \nabla f(B_s)\cdot \mathrm{d}B_s + \frac{1}{2}\int_0^t \Delta f(B_s)\mathrm{d}s. \end{aligned}$$

OBS: $f(B_t)$ is a martingale only if $\Delta f = 0$.

7 **a)** We apply Itô's formula to the function $g(t,x) = e^{t/2}\cos x \in C^2([0,+\infty)\times\mathbb{R})$,

$$\begin{aligned} dX_t &= d(e^{t/2}\cos B_t) \\ &= \frac{1}{2}e^{t/2}\cos B_t dt - e^{t/2}\sin B_t dB_t - \frac{1}{2}e^{t/2}\cos B_t\ dt \\ &= -e^{t/2}\sin B_t\ dB_t. \end{aligned}$$

Note that $h(s,\omega) := e^{t/2}\sin B_t$ is $\mathcal{B}\times\mathcal{F}$ measurable and $\{\mathcal{F}_t\}_t$-adapted, since and $B_t(\omega)$ is and $e^{t/2}\sin(x)$ is continuous. Moreover, for any $t \geq 0$,

$$\begin{aligned} E\left(\int_0^t h(s,\cdot)^2 ds\right) &= E\left(\int_0^t e^s \sin^2(B_s)ds\right) \\ &\leq E\left(\int_0^t e^s ds\right) \\ &= e^t - 1 < +\infty. \end{aligned}$$

Hence $h(s,\omega) \in \mathcal{V}(0,t)$. By Corollary 3.2.6 in Øksendal, it then follows that

$$X_t(\omega) = X_0 - \int_0^t e^{s/2}\sin B_s\ dB_s$$

is a martingale w.r.t. $\mathcal{F}_t$.

b) We apply Itô's formula to the function $g(t,x) = e^{t/2}\sin x \in C^2([0,+\infty)\times\mathbb{R})$,

$$\begin{aligned} dX_t &= d(e^{t/2}\sin B_t) \\ &= \frac{1}{2}e^{t/2}\sin B_t dt + e^{t/2}\cos B_t\ dB_t - \frac{1}{2}e^{t/2}\sin B_t\ dt \\ &= e^{t/2}\cos B_t\ dB_t. \end{aligned}$$

We can verify that the function $h(s,\omega) := e^{s/2}\cos(B_s(\omega)) \in \mathcal{V}(0,t)$ for any finite $t \geq 0$ as in part a). By Corollary 3.2.6 in Øksendal, it then follows that

$$X_t(\omega) = X_0 - \int_0^t e^{s/2}\cos B_s dB_s$$

is a martingale w.r.t. $\mathcal{F}_t$.

c) For the function $g(t,x) = (x+t)\exp(-x-t/2) \in C^2\left([0,+\infty)\times\mathbb{R}\right)$, we compute

$$\begin{aligned}\frac{\partial g}{\partial t} &= \left(1-\frac{t+x}{2}\right)\exp\left(-x-t/2\right),\\ \frac{\partial g}{\partial x} &= (1-t-x)\exp\left(-x-t/2\right),\\ \frac{\partial^2 g}{\partial x^2} &= (-2+t+x)\exp\left(-x-t/2\right),\end{aligned}$$

and apply Itô's formula,

$$\begin{aligned}dX_t &= dg(t,B_t)\\ &= \left(\left(1-\frac{t+B_t}{2}\right)dt + (1-t-B_t)\,dB_t - \left(1-\frac{t+B_t}{2}\right)dt\right)\exp\left(-B_t-t/2\right)\\ &= (1-t-B_t)\exp\left(-B_t-t/2\right)dB_t.\end{aligned}$$

Now we verify that $h(t,\omega) := (1-t-B_t(\omega))\exp(-B_t(\omega)-t/2) \in \mathcal{V}(0,t)$. As in a), the joint measurability and $\{\mathcal{F}_t\}_t$-adaptedness is evident. Note that

$$\begin{aligned}E(\exp(aB_s)) &= \int_{\mathbb{R}} e^{ax}\frac{1}{\sqrt{2\pi s}}\exp\left(\frac{-x^2}{2s}\right)dx = \exp(a^2 s/2), \text{ for all } a\in\mathbb{R},\\ x^2\exp(-2x) &\le 1+\exp(-4x), \text{ for all } x\in\mathbb{R}.\end{aligned}$$

In particular, $E(\exp(-2B_s)) = \exp(2s)$, and

$$E\left(B_s^2\exp(-2B_s)\right) \le E\left(1+\exp(-4B_s)\right) = 1+\exp(8s).$$

(You can also estimate the above expectation by Cauchy-Schwartz inequality.) So for all $t\ge 0$,

$$\begin{aligned}E\left(\int_0^t h(s,\cdot)^2 ds\right) &= E\left(\int_0^t (1-s-B_s)^2\exp(-2B_s-s)ds\right)\\ &\le 2E\left(\int_0^t \left((1-s)^2+B_s^2\right)\exp(-2B_s-s)ds\right)\\ &= 2\int_0^t \exp(-s)\left((1-s)^2 E\left(\exp(-2B_s)\right) + E\left(B_s^2\exp(-2B_s)\right)\right)ds\\ &\le 2\int_0^t \exp(-s)\left((1-s)^2\exp(2s)+1+\exp(8s)\right)ds\\ &= 2\int_0^t \left((1-s)^2\exp(s)+1+\exp(7s)\right)ds\\ &< +\infty.\end{aligned}$$

Therefore $h(s,\omega)\in\mathcal{V}(0,t)$, and by Corollary 3.2.6 in Øksendal,

$$X_t(\omega) = X_0 + \int_0^t (1-s-B_s)\exp\left(-B_s-s/2\right)dB_s$$

is a martingale w.r.t. $\mathcal{F}_t$.

9 Since X_t is an Ito process, $u \in \mathcal{W}$, and since u is also bounded, $|u(t,\omega)| \leq C < \infty$ for a.e. (t,ω), it follows that $u(t,\omega) \in \mathcal{V}[0,T]$. By *Excercise* 4.4 in Øksendal with $n = 1, \theta = -u$, M_t is a martingale and

$$dM_t = -M_t u(t,\omega) dB_t.$$

Since $dX_t = udt + dB_t$, we then use Ito product rule (*Excercise* 4.3 in Øksendal) to find that

$$\begin{aligned} dY_t &= X_t dM_t + M_t dX_t + dX_t dM_t \\ &= X_t(-M_t u dB_t) + M_t(udt + dB_t) + (udt + dB_t)(-M_t u dB_t) \\ &= -uM_t X_t dB_t + uM_t dt + M_t dB_t - uM_t dt \\ &= M_t(1 - uX_t) dB_t. \end{aligned} \tag{2}$$

By Corollary 3.2.6 in Øksendal, Y_t is a martingale if

$$V_t(\omega) := M_t(1 - uX_t) \in \mathcal{V}[0,T].$$

First, by the general Ito formula (Theorem 4.2.1 in Øksendal) applied to $g(t,(x_1,x_2)) = x_1 x_2$, $Y_t = X_t M_t$ is an Ito process since X_t, M_t are Ito processes (on the same filtered probability space). Hence, by (2), $V_t(\omega) \in \mathcal{W}$ and in particular, it is $\mathcal{B} \times \mathcal{F}$-measurable and $\mathcal{F}_t$-adapted.

By the hint, $X_t, M_t \in L^p(\Omega)$ and $E(|X_t|^p) < \infty$ and $E(|M_t|^p) < \infty$ for all $p \in [1,\infty)$, so by (Fubini and) the Cauchy-Schwartz inequality,

$$\begin{aligned} \|V_s\|^2_{L^2([0,T]\times\Omega)} &= \int_0^T E\left((V_s(\omega))^2 \right) ds \\ &= \int_0^T E\left| M_s^2 (1 - uX_s)^2 \right| ds \\ &\leq \int_0^T \sqrt{E(M_s^4) E((1 - uX_s)^4)} ds \\ &\leq \int_0^T \sqrt{8E(M_s^4)\left(1 + C^4 E|X_s|^4\right)} ds \\ &< \infty. \end{aligned}$$

Hence, $V_t(\omega) = M_t(1 - uX_t) \in \mathcal{V}[0,T]$ and Y_t is an $\mathcal{F}_t$-martingale.

10 **(i)** We will show that $X_t = e^{B_t}$ solves

$$dX_t = \frac{1}{2} X_t dt + X_t dB_t.$$

First note that $dB_t = 0dt + 1dB_t$ is an Itô process. Let $g(t,x) = e^x$ (a C^2-function) and $X_t = g(t, B_t)$, and use the Itô Formula (Theorem 4.1.2 in Øksendal) to calculate

$$\begin{aligned} dX_t &= g_t|_{t,B_t} dt + g_x|_{t,B_t} dB_t + \frac{1}{2} g_{xx}|_{t,B_t} \cdot 1^2 dt \\ &= 0 + e^x|_{B_t} dB_t + \frac{1}{2} e^x|_{B_t} dt \\ &= e^{B_t} dB_t + \frac{1}{2} e^{B_t} dt \\ &= X_t dB_t + \frac{1}{2} X_t dt \end{aligned}$$

where the vertical lines denote "insert for t and x". By the Ito formula (the above theorem), the process $X_t = g(t, B_t)$ is again an Itô process and it satisfies the SDE a.e. Hence X_t is a a strong solution of the SDE.

(iv) We will show that $\mathbf{X_t} = (X_1(t), X_2(t)) = (t, e^t B_t)$ solves

$$d\mathbf{X_t} = \begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^{X_1} \end{bmatrix} dB_t \tag{3}$$

Here it looks like we are supposed to use the general Ito's formula. But since the first equation is trivial, we only need the 1D Ito formula for the second equation. Let first $g_1(t, x) = t$ and $X_1(t) = t$, then from usual calculus,

$$dX_1 = 1dt.$$

Next, let $g(t, x) = e^t x$ and $X_2(t) = g(t, B_t)$, then by the Ito formula,

$$\begin{aligned} dX_2 &= g_t|_{t,B_t} dt + g_x|_{t,B_t} dB_t + \frac{1}{2} g_{xx}|_{t,B_t} \cdot 1^2 dt \\ &= g(t, B_t)dt + e^t dB_t + 0 \\ &= X_2 dt + e^t dB_t. \end{aligned}$$

Since $X_1 = t$, it follows that X satisfy equation (3) a.s. Since we did not use the general Ito formula, we now have to check that X is an Ito process. Let $u = (1, X_2)$ and $v = (0, e^{X_1})$. Joint measurability and $\mathcal{F}_t$-adaptedness of X_t, u_t, v_t is obvious, and for all $t \geq 0$,

$$E \int_0^t (|u_s| + |v_s|^2) ds \leq \frac{1}{2} t^2 + e^t \int_0^t E|B_s|^2 ds = \frac{1}{2} t^2 (1 + e^t) < \infty.$$

We have shown that X is an Ito process satisfying (3) a.s. Hence it is a strong solution of (3).

11 This equation,

$$dX_t = rX_t dt + X_t \sum_{k=1}^{n} \alpha_k dB_k(t),$$

is somewhat similar to the equation in Example 5.1.1 in the sense that the growth coefficient,

$$r + \sum_{k=1}^{n} \alpha_k \frac{dB_k}{dt}(t),$$

is equal to a constant r plus random "white noise". It seems that the same trick as in Example 5.1.1 should be feasible, so we introduce the transformation

$$Y_t = g(X_t) = \log(X_t)$$

and obtain from Itô's Formula,

$$\begin{aligned} dY_t &= \frac{\partial g}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} (dX_t)^2 \\ &= \frac{1}{X_t} dX_t - \frac{1}{2} \frac{1}{X_t^2} (dX_t)^2. \end{aligned} \tag{4}$$

The squared differential $(dX_t)^2$ may be evaluated according to the "rules",

$$dt \cdot dt = dt \cdot dB_k(t) = 0, \; dB_k(t)\, dB_l(t) = \delta_{kl} dt,$$

and hence,

$$(dX_t)^2 = X_t^2 \sum_{k=1}^{n} \alpha_k^2 dt.$$

If this inserted into Eqn. (4), we obtain

$$dY_t = \left(r - \frac{1}{2} \sum_{k=1}^{n} \alpha_k^2 \right) dt + \sum_{k=1}^{n} \alpha_k dB_k(t).$$

Since $Y_t = \ln X_t$, it follows that

$$X_t = X_0 \exp \left[\left(r - \frac{1}{2} \sum_{k=1}^{n} \alpha_k^2 \right) t + \sum_{k=1}^{n} \alpha_k B_k(t) \right],$$

where we have assumed $B_k(0) = 0$.