Norwegian University of Science and Technology
Department of Mathematical Sciences

MA8109 Stochastic Processes in Systems Theory Autumn 2013

Exercise set 4 – solutions

1 (MA8109 Exam 2003, problem 3b) This is a linear equation of the form

$$dX_t = p(t) X_t dt + q(t) dB_t,$$

which is generally solvable using an integrating factor:

$$h(t) dX_t = d(h(t) X_t) - h'(t) X_t dt = \frac{h(t)}{t} dX_t + h(t) t dB_t.$$

Choose h such that $-h'(t) = h(t)/t$, e.g. $h(t) = t^{-1}$. Then

$$d\left(\frac{X_t}{t}\right) = dB_t$$

so that

$$\frac{X_t}{t} - \frac{1}{1} = B_t - B_1.$$

The solution may then be written

$$X_t = t(1 + B_t - B_1) = t\tilde{B}_t^{1,1}, \ t \geq 1.$$

Here, $\tilde{B}_t^{1,1}$ means a regular BM starting at $x = 1$ for $t = 1$.

2 **(i)** This is a system of equations, which may be written out as

$$dX_1(t) = dt + dB_1(t),$$
$$dX_2(t) = X_1(t) dB_2(t). \tag{1}$$

Let us assume that the initial values are given by $X_1(0) = x_1$ and $X_2(0) = x_2$, and as usual, $B_1(0) = B_2(0) = 0$, where B_1 and B_2 are *independent* standard Brownian motions.

We observe that the first equation is completely independent of X_2 and B_2, and may therefore be solved right away,

$$X_1(t) = x_1 + t + B_1(t). \tag{2}$$

This gives us the second equation as

$$dX_2(t) = [x_1 + t + B_1(t)] dB_2, \tag{3}$$

which may also be solved by a simple integration,

$$X_2(t) = x_2 + \int_0^t (x_1 + s + B_1(s)) dB_2(s)$$
$$= x_2 + x_1 B_2(t) + \int_0^t (s + B_1(s)) dB_2(s), \tag{4}$$

and this is the solution stated in B.Ø.

There are several alternate forms of the solution. First of all,

$$\int_0^t sdB_2(s) = tB_2(t) - \int_0^t B_2(s)\,ds \tag{5}$$

(see, e.g. Exercise 3.1). Thus,

$$X_2(t) = x_2 + (x_1 + t)\,B_2(t) - \int_0^t B_2(s)\,ds + \int_0^t B_1(s)\,dB_2(s). \tag{6}$$

It is also possible to apply the product formula (Exercise 4.3),

$$d\left[X_1(t)\,B_2(t)\right] = X_1(t)\,dB_2(t) + B_2(t)\,dX_1(t) + dX_1(t)\,dB_2(t). \tag{7}$$

The last term is, according to the rules, equal to 0,

$$dX_1(t)\,dB_2(t) = (dt + dB_1(t))\,dB_2(t) = 0. \tag{8}$$

Thus,

$$dX_2 = d\left[X_1(t)\,B_2(t)\right] - B_2(t)\,dX_1(t), \tag{9}$$

leading to

$$X_2(t) = x_2 + X_1(t)\,B_2(t) - \left(\int_0^t B_2(s)\,ds + \int_0^t B_2(s)\,dB_1(s)\right). \tag{10}$$

Show that this solution is the same as the one above!

(ii) and (iii)

Both these equations are of the form

$$dX_t = p(t)\,X_t dt + q(t)\,dB_t \tag{11}$$

and may be solved by introducing an integration factor, as discussed in the lecture. We multiply the equation by a function $h(t)$ and use Itô's Formula,

$$d\left[h(t)\,X_t\right] = \frac{dh}{dt}(t)\,X_t dt + h(t)\,dX_t, \tag{12}$$

so that the equation becomes

$$d\left[h(t)\,X_t\right] - \frac{dh}{dt}(t)\,X_t dt = h(t)\,p(t)\,X_t dt + h(t)\,q(t)\,dB_t. \tag{13}$$

The idea is then to choose h so that

$$-\frac{dh}{dt} = h(t)\,p(t), \tag{14}$$

and the resulting equation becomes

$$d\left(h(t)\,X_t\right) = h(t)\,q(t)\,dB_t, \tag{15}$$

with the solution

$$X_t = \frac{X_0 h(0) + \int_0^t h(s)\,q(s)\,dB_s}{h(t)}. \tag{16}$$

For Exercise (ii), Eqn. 14 becomes

$$-\frac{dh}{dt} = h \tag{17}$$

and then

$$h(t) = e^{-t}, \tag{18}$$

as stated in the book. The solution is then

$$X_t = \frac{X_0 h(0) + \int_0^t e^{-s} dB_s}{e^{-t}} = X_0 e^t + \int_0^t e^{t-s} dB_s. \tag{19}$$

Similarly, for Exercise (iii), the equation for $h(t)$ is

$$-\frac{dh}{dt} = -h, \tag{20}$$

and $h(t) = e^t$. From Eqn. 16 we then obtain

$$X_t = \frac{X_0 + \int_0^t e^s e^{-s} dB_s}{e^t} = X_0 e^{-t} + e^{-t} B_t, \tag{21}$$

assuming $B_0 = 0$.

3 (Øksendal 5:5.5)

a) We multiply X_t by the integrating factor $e^{-\mu t}$, and then employ Itô's formula:

$$\begin{aligned} d(e^{-\mu t} X_t) &= d(e^{-\mu t}) X_t + e^{-\mu t} dX_t + d(e^{-\mu t}) dX_t \\ &= -\mu e^{-\mu t} X_t dt + e^{-\mu t}(\mu X_t dt + \sigma dB_t) - \mu e^{-\mu t} dt(\mu X_t dt + \sigma dB_t) \\ &= \sigma e^{-\mu t} dB_t. \end{aligned}$$

Hence $e^{-\mu t} X_t = X_0 + \int_0^t \sigma e^{-\mu s} dB_s$, and

$$X_t = e^{\mu t} X_0 + \int_0^t \sigma e^{\mu(t-s)} dB_s.$$

By Ito's formula with $f(s,x) = e^{\mu(t-s)} x$ ("integration by parts"), we can simplify further since

$$\int_0^t e^{\mu(t-s)} dB_s = B_t + \mu \int_0^t e^{\mu(t-s)} B_s ds.$$

b) For the stochastic process above, we have

$$E(X_t) = E(e^{\mu t} X_0) + E\left(\int_0^t \sigma e^{\mu(t-s)} dB_s\right) = e^{\mu t} E(X_0),$$

since the Itô integral is a martingale when the integrand belongs to $\mathcal{V}[0,t]$ (ok since $\int_0^t \left(e^{\mu(t-s)}\right)^2 ds < \infty$ for any $t > 0$).

If we assume X_0 is independent of the Brownian Motion B_t,

$$\begin{aligned} Var(X_t) &= Var(e^{\mu t}X_0 + \int_0^t \sigma e^{\mu(t-s)} dB_s) \\ &= e^{2\mu t}Var(X_0) + Var\left(\int_0^t \sigma e^{\mu(t-s)} dB_s\right) \\ &= e^{2\mu t}Var(X_0) + E\left(\left(\int_0^t \sigma e^{\mu(t-s)} dB_s\right)^2\right) \\ &= e^{2\mu t}Var(X_0) + E\left(\int_0^t \sigma^2 e^{2\mu(t-s)} ds\right) \\ &= e^{2\mu t}Var(X_0) + \frac{\sigma^2}{2\mu}(e^{2\mu t} - 1). \end{aligned}$$

4 It is stated in the exercise that also this equation may be solved by an integrating factor multiplying both sides,

(22) $$F_t dY_t = F_t\left(rdt + \alpha Y_t dB_t\right).$$

However, here the dependent variable Y_t is in the dB_t-term, and the factor is already given in the problem.

Let us, nevertheless, try to find the integration factor directly by assuming that it is of the form

(23) $$F_t = f\left(t, B_t\right).$$

From Itô's Formula we have

(24) $$dF_t = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \frac{\partial f}{\partial x}dB_t,$$

and the idea for the integrating factor is again to make use of the product formula,

(25) $$d\left(F_tY_t\right) = F_t dY_t + Y_t dF_t + dF_t dY_t.$$

In the present case,

(26) $$dF_t dY_t = \left[\left(\frac{\partial f}{\partial t} + + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \frac{\partial f}{\partial x}dB_t\right]\cdot\left(rdt + \alpha Y_t dB_t\right) = \frac{\partial f}{\partial x}\alpha Y_t dt$$

Inserting this and the equation itself into Eqn. 25 leads to

$$\begin{aligned} d\left(F_tY_t\right) &= f\cdot\left(rdt + \alpha Y_t dB_t\right) + Y_t\left(\left(\frac{\partial f}{\partial t} + + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \frac{\partial f}{\partial x}dB_t\right) + \alpha Y_t\frac{\partial f}{\partial x}dt \\ &= \left(fr + Y_t\left(\frac{\partial f}{\partial t} + \alpha\frac{\partial f}{\partial x} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right)\right)dt + \left(f\alpha + \frac{\partial f}{\partial x}\right)Y_t dB_t \end{aligned} \tag{27}$$

The equation would be easy to solve if we were able to find some function f such that

$$\begin{aligned} f\alpha + \frac{\partial f}{\partial x} &= 0, \\ \frac{\partial f}{\partial t} + \alpha\frac{\partial f}{\partial x} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} &= 0. \end{aligned} \tag{28}$$

Any such function will do, and the first equation is always satisfied for

$$f(t,x) = g(t)\, e^{-\alpha x}. \tag{29}$$

Putting this into the second leads to

$$\frac{dg}{dt} + \left(-\alpha^2 + \frac{1}{2}\alpha^2\right) g = \frac{dg}{dt} - \frac{1}{2}\alpha^2 g = 0, \tag{30}$$

which is satisfied for

$$g(t) = e^{\alpha^2 t/2}. \tag{31}$$

We may therefore use the integrating factor

$$F_t = \exp\left(\frac{\alpha^2}{2}t - \alpha B_t\right), \tag{32}$$

which turns out to be the same as the one given in the book.

Eqn. 27 now simplifies to

$$d(F_t Y_t) = r F_t dt, \tag{33}$$

with the solution

$$F_t Y_t - F_0 Y_0 = \int_0^t r F_s ds, \tag{34}$$

or

$$Y_t = \exp\left(-\frac{\alpha^2}{2}t + \alpha B_t\right)\left(Y_0 + \int_0^t r \exp\left(\frac{\alpha^2}{2}s - \alpha B_s\right) ds\right) \tag{35}$$

5 (Øksendal 5.5.7) By setting $Y_t = X_t - m$, the equation

$$dX_t = (m - X_t)\, dt + \sigma dB_t$$

is turned into

$$dY_t = -Y_t dt + \sigma dB_t.$$

Here the integrating factor is $h(t) = e^t$ since

$$d\left(e^t Y_t\right) - e^t Y_t dt = e^t dY_t = -e^t Y_t dt + \sigma e^t dB_t,$$

and thus

$$e^t Y_t - Y_0 = \sigma \int_0^t e^s dB_s$$

or

$$X_t = Y_t + m = m + (X_0 - m)\, e^{-t} + \sigma \int_0^t e^{s-t} dB_s$$

Since the Itô integral has expectation 0 and variance given by the Itô isometry, we have, for X_0 independent of B_t,

$$\begin{aligned} \mathsf{E}X_t &= m + (\mathsf{E}X_0 - m)\, e^{-t}, \\ \operatorname{Var} X_t &= e^{-2t} \operatorname{Var} X_0 + \sigma^2 \int_0^t e^{2(s-t)} ds = e^{-2t} \operatorname{Var} X_0 + \frac{\sigma^2}{2}\left(1 - e^{-2t}\right). \end{aligned}$$

We note that for $t \to \infty$, $\mathsf{E}X_t \to m$ and $\operatorname{Var} X_t \to \frac{\sigma^2}{2}$.

6* (Øksendal 5:5.9) We use Theorem 5.2.1 in Øksendal to conclude that there is a unique strong solution of

$$\begin{cases} \mathrm{d}X_t = \ln(1+X_t^2)\mathrm{d}t + \chi_{\{X_t>0\}}X_t\mathrm{d}B_t & (t,\omega)\in[0,\infty)\times\mathbb{R} \\ Z := X_0 = a & \omega\in\mathbb{R}, \end{cases} \tag{36}$$

where $a\in\mathbb{R}$ and χ is the characteristic function.

We set $b(t,X_t) = \ln(1+X_t^2)$ and $\sigma(t,X_t) = \chi_{\{X_t>0\}}X_t$, and check below that the assumptions ((A1)–(A3)) of the theorem are satisfied.

(A1) $b(t,X_t)$ and σ are Lipschitz continuous: By the Mean Value Theorem

$$\left|\frac{b(t,x)-b(t,y)}{x-y}\right| = \left|\frac{\ln(1+x^2)-\ln(1+y^2)}{x-y}\right| \overset{\text{MVT}}{=} \left|\frac{\mathrm{d}}{\mathrm{d}x}\ln(1+x^2)\mid_{x=\xi}\right|$$
$$= \max_{x\in\mathbb{R}}\left|\frac{2x}{1+x^2}\right| \le 1,$$

and

$$|\sigma(t,x)-\sigma(t,y)| = |\chi_{\{x>0\}}x - \chi_{\{y>0\}}y| \le |x-y|.$$

Since b,σ are independent of t, measurability follows from continuity in x.

(A2) Since b,σ are independent of t, linaer growth is a direct consequence of the Lipschitz estimate (A1): By adding and subtracting terms and using the Lipschitz result,

$$|b(t,x)| \le |b(t,0)| + |b(t,x)-b(t,0)| = 0 + 1\cdot|x|,$$

and

$$|\sigma(t,x)| \le |\sigma(t,0)| + |\sigma(t,x)-\sigma(t,0)| = 0 + 1\cdot|x|$$

(A3) $Z=a$ is in L^2 since $E|a|^2 = |a|^2 < \infty$. Moreover, a is independent of $\mathcal{F}_\infty^{(m)} = \mathcal{F}_{\{B_s:s\ge 0\}}$ since a is a constant (constants are independent of any stochastic variable).

7* (Øksendal 5:5.10) (A slightly simpler argument than suggested in the hint). Suppose that b,σ,Z are as in theorem 5.2.1. Then the stochastic differential equation

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t$$

has a unique strong solution X_t for $t\in[0,T]$ satisfying $X_0 = Z$. We wish to find a bound on $E(\|X_t\|^2)$. By the Ito's formula we have

$$\begin{aligned} d\|X_t\|^2 &= 2\langle X_t, dX_t\rangle + \langle dX_t, dX_t\rangle \\ &= 2\langle X_t, b(t,X_t)\rangle dt + 2\langle X_t, \sigma(t,X_t)dB_t\rangle + \operatorname{tr}(\sigma(t,X_t)^T\sigma(t,X_t))dt \\ &= (2\langle X_t, b(t,X_t)\rangle + \|\sigma(t,X_t)\|^2)dt + 2\langle X_t, \sigma(t,X_t)dB_t\rangle, \end{aligned}$$

whence

$$E(\|X_t\|^2) = E(\|Z\|^2) + E\left[\int_0^t (2\langle X_s, b(s,X_s)\rangle + \|\sigma(s,X_s)\|^2)ds\right].$$

Now, by the assumptions, we have

$$\begin{aligned} 2\langle X_s, b(t, X_s)\rangle + \|\sigma(t, X_s)\|^2 &\le \|X_s\|^2 + \|b(s, X_s)\|^2 + \|\sigma(s, X_s)\|^2 \\ &\le \|X_s\|^2 + (\|b(s, X_s)\| + \|\sigma(s, X_s)\|)^2 \\ &\le \|X_s\|^2 + C^2(1 + \|X_s\|)^2 \\ &\le \|X_s\|^2 + 2C^2(1 + \|X_s\|^2) \\ &= 2C^2 + (1 + 2C^2)\|X_s\|^2, \end{aligned}$$

and so

$$E(\|X_t\|^2) \le E(\|Z\|^2) + 2C^2 t + (1 + 2C^2) \int_0^t E(\|X_s\|^2) ds.$$

Grönwall's lemma now implies that

$$E(\|X_t\|^2) \le (E(\|Z\|^2) + 2C^2 t) e^{(1+2C^2)t}.$$

8 The equation has a form we have not really addressed,

$$dY_t = \frac{b - Y_t}{1 - t} dt + dB_t,$$

but by writing

$$X_t = b - Y(t), \tag{37}$$

we obtain the *linear* SDE

$$dX_t = \frac{1}{t - 1} X_t dt - dB_t, \ X_0 = b - a. \tag{38}$$

This equation may, as before, be solved by an integrating factor (when $0 < t < 1$). The solution is

$$X_t = \frac{X_0 h(0) + \int_0^t h(s)(-1)\, dB_s}{h(t)}, \tag{39}$$

where h satisfies

$$-\frac{dh}{dt} = \frac{h}{t - 1}, \tag{40}$$

e.g. $h(t) = \frac{1}{t-1}$. Then

$$\begin{aligned} X_t &= (t - 1)\left((-1) - \int_0^t \frac{dB_s}{s - 1}\right) \\ &= (b - a)(1 - t) - (1 - t)\int_0^t \frac{dB_s}{1 - s} \end{aligned} \tag{41}$$

or

$$Y_t = b - X_t = a(1 - t) + bt + (1 - t)\int_0^t \frac{dB_s}{1 - s}, \tag{42}$$

which is the solution stated in the book.

This solves the equation for us, although the exercise only requires that this solution is *verified*. Putting the solution into the left hand side of the equation gives

$$LHS = dY_t = (-a+b)\,dt - dt \cdot \int_0^t \frac{dB_s}{1-s} + (1-t)\,\frac{1}{(1-t)} dB_t \tag{43}$$

$$= (-a+b)\,dt + dB_t - dt \cdot \int_0^t \frac{dB_s}{1-s}. \tag{44}$$

Similarly, for right hand side,

$$RHS = \frac{b-Y_t}{1-t}dt + dB_t = \frac{b - \left(a\,(1-t) + bt + (1-t)\int_0^t \frac{dB_s}{1-s}\right)}{1-t}dt + dB_t$$

$$= (b-a)\,dt + dB_t - dt \cdot \int_0^t \frac{dB_s}{1-s}, \tag{45}$$

and $LHS = RHS$.

The limit when $t \to 1$ is not at all obvious, and requires that

$$\lim_{t \to 1-} (1-t) \int_0^t \frac{dB_s\,(\omega)}{1-s} = 0 \ a.s. \tag{46}$$

Let

$$g\,(t,\omega) = (1-t) \int_0^t \frac{dB_s\,(\omega)}{1-s}. \tag{47}$$

It is easy to see, from the *Itô Isometry* (and $E\left(\int_0^t \frac{dB_s(\omega)}{1-s}\right) = 0$) that

$$\begin{aligned} \operatorname{Var}\left(g\,(t,\omega)\right) &= E\left(g\,(t,\omega)^2\right) \\ &= \|g\,(t)\|^2_{L^2(\Omega)} && (48) \\ &= (1-t)^2 \int_0^t \frac{ds}{(1-s)^2} && (49) \\ &= (1-t)^2 \left[\frac{1}{1-s}\right]_0^t = (1-t)^2 \left(\frac{1}{1-t} - 1\right) = t\,(1-t)\,. \end{aligned}$$

Thus, $\|g\,(t)\|_{L^2(\Omega)} \underset{t\to 1}{\longrightarrow} 0$, but this is not quite sufficient for the limit in Eqn. 46. However, it is now at least possible to find a sequence $\{t_k\}$ such that $t_k \to 1$ and $\lim_{k\to\infty} g\,(t_k,\omega) = 0\ a.s.$ Which means that if it converges at all, it has to be to 0.

This is probably as far as we are able to come by "elementary" means.

The book gives us a hint of applying the *Doob's Martingale Inequality* stated (without proof) in Thm. 3.2.4. Following B.Ø., we let the sequence $\{t_n\}$ be defined $t_n = 1 - 2^{-n}$, $I_n = [t_n, t_{n+1})$, and set

$$M_t = \int_0^t \frac{dB_s}{1-s}. \tag{50}$$

Note that M_t, since it is defined in terms of an Itô integral, is an L^2-martingale. Note also that all I_n are disjoint and that the union makes up the whole of $[0,1)$.

Then

$$
\begin{aligned}
P\left(\omega\ ;\ \sup_{t\in I_n}\left|(1-t)\,M_t(\omega)\right|>\varepsilon\right) &< P\left(\omega\ ;\ (1-t_n)\sup_{t\in I_n}\left|M_t(\omega)\right|>\varepsilon\right)\\
&< P\left(\omega\ ;\ (1-t_n)\sup_{0<t<t_{n+1}}\left|M_t(\omega)\right|>\varepsilon\right)\\
&= P\left(\omega\ ;\ \sup_{0<t<t_{n+1}}\left|M_t(\omega)\right|>\frac{\varepsilon}{(1-t_n)}\right)\\
&\le \frac{(1-t_n)^2}{\varepsilon^2}E\left(M_{t_{n+1}}^2\right) \qquad (51)\\
&= \frac{(1-t_n)^2}{\varepsilon^2}\times\frac{1}{1-t_{n+1}}\\
&= 2^{-2n}\frac{1}{\varepsilon^2}2^{n+1} = 2\varepsilon^{-2}2^{-n},
\end{aligned}
$$

which is the stated inequality. The trick is now to consider

$$A_n = \left\{\omega\ ;\ \sup_{t\in I_n}\left|(1-t)\,M_t(\omega)\right|>2^{-n/4}\right\} = \left\{\omega\ ;\ \sup_{t\in I_n}\left|g(t,\omega)\right|>2^{-n/4}\right\}. \qquad (52)$$

From the inequality above we obtain that

$$P(A_n) < 2\left(2^{-n/4}\right)^{-2}2^{-n} = 2\cdot 2^{-n/2}, \qquad (53)$$

and hence

$$\sum_{n=1}^{\infty}P(A_n)<\infty. \qquad (54)$$

The rest is piece of cake using (the easy part of) the *Borel-Cantelli Lemma*: For almost all paths, $g(t,\omega)$ there exists an $N(\omega)$ such that $\omega\notin A_n$ *for all* $n>N(\omega)$. Thus, in that case,

$$\sup_{t\in I_n}\left|g(t,\omega)\right|\le 2^{-n/4} \qquad (55)$$

for all $n>N(\omega)$. We finally observe that $\cup_{n=N(\omega)}^{\infty}I_n=[1-2^{-N(\omega)},1)$, so that, indeed,

$$\lim_{t\to 1}\left|g(t,\omega)\right|=0 \qquad (56)$$

for these cases, that is, *almost surely.*