Norwegian University of Science and Technology
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MA8109 Stochastic Processes in Systems Theory Autumn 2013

Exercise set 5

1 Øksendal Exercise 7:7.1

2 Øksendal Exercise 7:7.2

3 Øksendal Exercise 7:7.8

4 Øksendal Exercise 7:7.18

5 (Øksendal 7:7.9) Let X_t^x be a geometric Brownian motion, i.e.,

$$dX_t = rX_t dt + \alpha X_t dB_t, \qquad X_0 = x > 0,$$

where $B_t \in \mathbb{R}$ and r, α are constants.

a) Find the generator A of X_t^x and compute $Af(x)$ when $f(x) = x^\gamma$ and $x > 0$ and γ is constant.

b) If $r < \frac{1}{2}\alpha^2$, then $X_t \to 0$ as $t \to \infty$, a.s. (see Example 5.1.1 in Øksendal). What is the probability p that X_t^x, starting from $x < R$ ever hits the value R? Use Dynkin's formula with $f(x) = x^{\gamma_1}$ for $\gamma_1 = 1 - \frac{2r}{\alpha^2}$, to prove that

$$p = \left(\frac{x}{R}\right)^{\gamma_1}.$$

c) If $r > \frac{1}{2}\alpha^2$, then $X_t^x \to \infty$ as $t \to \infty$, a.s. Let

$$\tau = \inf\{t > 0 : X_t^x \geq R\}.$$

Use Dynkin's formula with $f(x) = \ln x$, $x > 0$, to prove that

$$E(\tau) = \frac{\ln \frac{R}{x}}{r - \frac{1}{2}\alpha^2}.$$

Hints:

b) Note that $\gamma_1 \in (0, 1)$. Consider

$$\tau_\rho = \inf\{t > 0 : X_t^x \in (\rho, R)^c\} \qquad \text{and} \qquad \tau_{\rho,k} = \min(\tau_\rho, k).$$

Since $r < \frac{1}{2}\alpha^2$, $X_t^x \to 0$ a.s. as $t \to \infty$ (see Example 5.1.1 in Øksendal),

$$\tau_\rho < \infty \quad a.s.$$

Use Dynkin's formula with $\tau_{\rho,k}$, send first $k \to \infty$, and then $\rho \to 0$. You may assume that $p(\rho) = P(\omega : X_{\tau_\rho}^x = R) \to P(\omega : X_{\tau_0}^x = R) = p$.

c) Note that $\gamma_1 < 0$ and $\tau < \infty$ a.s. Consider $\tau_{\rho,k}$, send $k \to \infty$ and then $\rho \to 0$. You need estimates for

$$(1 - p(\rho)) \ln \rho,$$

where

$$p(\rho) = P(X_t^x \text{ reaches the value } R \text{ before } \rho) = P(X_{\tau_\rho}^x = R),$$

which you can get from the calculations in a) and b).

6 We consider Brownian motion in $\mathbb{R}^n$, $B_t^x = x + B_t$.

a) Show that the probability of B_t^x of hitting a half space H in $\mathbb{R}^n$, starting from a point $x \notin H$, is 1 regardless of n. (Thus, we do not have the same situation as with finite size balls in $\mathbb{R}^n$).
Determine $\mathsf{E}\left(\tau_H^b\right)$ when $b \notin H$.

b) Determine a simple set in $\mathbb{R}^4$ with infinite measure so that the probability of hitting the set, starting outside it, is strictly smaller than 1.
(Hint: $B_t^{(4)} = \left\{B_t^{(3)}, B_t^{(1)}\right\}$).
What about an example working for $\mathbb{R}^3$?

c) Find an open set U in $\mathbb{R}^3$ with a finite volume and a hitting time τ_U so that $\mathsf{E}\left(\tau_{\bar{U}}^x\right) \leq 1$ for all $x \notin \bar{U}$ ($\bar{U}$ denotes the closure of U).

7 Øksendal Exercise 8:8.6

This problem was more difficult and less relevant than first expected. Here are detailed hints:

1. Consider a sequence of functions $f_k \in C_c^2$ such that

$$|f_k| \leq \|f\|_{L^\infty}, \qquad |Df_k| \leq \|Df\|_{L^\infty},$$

$$f_k(x) \to \tilde{f}(x) := (x - K)^+ \qquad \text{and} \qquad Df_k(x) \to D\tilde{f}(x) \quad \text{a.e.},$$

and go to the limit. (You can assume this sequence exists, it can be defined e.g. by mollification/convolution with a smooth kernel).

2. Assume $u(x, 0) = f_k(x) \in C_c^2$ and use the Feynman-Kac formula. The solution $u = u_k$ then satisfy the PDE.

3. Check that these solutions have similar integral form as the solution in the problem text and can be written as a convolution

$$u_k(x, t) = \int F_k(t, y) e^{-\frac{(y - \beta^{-1} \ln x)^2}{4t}} dy = \int F_k(t, y + \beta^{-1} \ln x) e^{-\frac{y^2}{4t}} dy.$$

4. Check that $u_k, \partial_t u_k, \partial_x u_k, \partial_x^2 u_k$ converge pointwisely to $u, \partial_t u, \partial_x u, \partial_x^2 u$ for $t > 0$ and $x > 0$ (the solution is not C^2 at $x = 0$).
Hint: Take one derivative on F and one on the Gauss kernel. When $x > 0$, the integrands will be bounded functions of y, use dominated convergence.

5. Go to the limit in the PDEs for u_k for $t > 0$ and $x > 0$. Can you show that the PDE holds also at the boundary $x = 0$?

8 In an uncorrelated Heston model, an improvement of the Black Scholes model in finance, the stock price S_t and volatility V_t are stochastic processes satisfying

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dB_{1,t},$$
$$dV_t = \alpha(\theta - V_t)dt + \beta\sqrt{V_t} dB_{2,t},$$

where $\mu, \alpha, \theta, \beta > 0$ are constants and $B = (B_1, B_2)$ is 2d B.M.

Let $(S_s^{t,x,v}, V_t^{t,x,v})$ be the solution of the SDE with initial conditions

$$(S_t, V_t) = (x, v).$$

The price of an option on the stock is given by the formula

$$u(x, v, t) = E(e^{-r(T-t)} f(S_T^{t,x,v})),$$

where f is a given function, e.g. $f(x) = (x - 1)^+$.

What partial differential equation and terminal value problem is satisfied by u when $f \in C_c^2(\mathbb{R})$?

Hint: $s = T - t$ and you may assume the Feynman-Kac formula applies.