MA8109
Stochastic Processes in Systems Theory
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Exercise set 5 – Solutions

1 Øksendal Exercise 7:7.1. See *Solutions and additional hints to some of the exercises* at the end of Øksendal.

2 Øksendal Exercise 7:7.2. See *Solutions and additional hints to some of the exercises* at the end of Øksendal.

3

a) Let $h(\omega) = (\tau_1 \vee \tau_2)(\omega)$. Then $\{\omega : h \leq t\} = h^{-1}([0,t]) = \tau_1^{-1}[0,t] \cup \tau_2^{-1}[0,t] \in N_t$ since N_t is a sigma algebra and τ_1, τ_2 are stopping times wrt N_t.
$h(\omega) = (\tau_1 \wedge \tau_2)(\omega)$ gives $h^{-1}[0,t] = \tau_1^{-1}[0,t] \cap \tau_2^{-1}[0,t] \in N_t$.

b) Let $h = \inf \tau_n = \lim \tau_n$. Then h is N_t measurable because $h^{-1}[0,t] = \bigcap_{n=1}^{\infty} \tau_n^{-1}[0,t]$ which is measurable because a sigma algebra is closed under countable intersection.

c) τ_U is a stopping time for every open set U by Øksendal Example 7.2.2. Let U_n be a decreasing sequence of open sets such that $F = \cap_n U_n$. Then τ_{U_n} is a decreasing sequence of stopping times and $\lim \tau_{U_n} = \tau_F$. Hence τ_F is a stopping time by part b).

4

a) We want to use Dynkin's formula to find $E[f(X_\tau^x)]$, but this requires that $f \in C_c^2$ and $E(\tau) < \infty$. The first requirement can be dealt with by replacing f by a function in C_c^2 which agrees with f on (a,b). The second requirement is handled by letting $\tau_k = \min(\tau, k)$ for $k \in \mathbb{N}$; then $E(\tau_k)$ is finite and we can apply Dynkin to get

$$\begin{aligned} E[f(X_{\tau_k}^x)] &= f(x) + E\left[\int_0^{\tau_k} \mathcal{A}f(X_s^x)\, ds\right] \\ &= f(x). \end{aligned}$$

We can now take the limit[1] to get that $E[f(X_\tau^x)] = \lim_{k\to\infty} E[f(X_{\tau_k}^x)] = f(x)$. Since

$$E[f(X_\tau^x)] = pf(b) + (1-p)f(a),$$

we can solve for p, to find that[2]

$$p = \frac{f(x) - f(a)}{f(b) - f(a)}. \tag{1}$$

[1] The technical details: use the dominated convergence theorem to pull the limit inside the expectation, and the continuity of f together with the t-continuity of X_t to get from τ_k to τ.
[2] At least assuming that we have a solution f with $f(b) - f(a) \neq 0$.

b) The equation $\mathcal{A}f = 0$ now reduces to $\frac{1}{2}f'' = 0$, which means that $f(s) = Cs + x$ for some constant C, where x is the initial condition. Inserting this into equation (1), we find

$$p = \frac{x-a}{b-a}.$$

c) $\mathcal{A}f = 0$ in this case becomes $\mu f' + \frac{\sigma^2}{2}f'' = 0$. The solution of this equation is given by $f(s) = C_1 e^{-\frac{2\mu}{\sigma^2}s} + C_2$, for some constants C_1 and C_2. Inserting this into equation (1), all constants cancel out, and we get

$$p = \frac{e^{-\frac{2\mu}{\sigma^2}x} - e^{-\frac{2\mu}{\sigma^2}a}}{e^{-\frac{2\mu}{\sigma^2}b} - e^{-\frac{2\mu}{\sigma^2}a}}.$$

5 This solution deviates somewhat from the hints in B.Ø.

a) From the *Generator Theorem* (Thm. 7.3.3) it follows that

$$A = rx\frac{d}{dx} + \frac{\alpha^2 x^2}{2}\frac{d^2}{dx^2}.$$

Thus,

$$A\left(x^\gamma\right) = \left(r\gamma + \frac{\alpha^2}{2}\gamma\left(\gamma - 1\right)\right)x^\gamma.$$

b) Here we can not assume that all paths hit the level $R > x$. Therefore, we consider the first exit time from the interval $[\rho, R]$ for $\rho < x$, say $\tau_{\rho,R}$. Following the hint, we take $f \in C_c^2(\mathbb{R})$ such that for $x \in [\rho, R]$,

$$f\left(x\right) = x^{\gamma_1} \qquad \text{with} \qquad \gamma_1 = 1 - \frac{2r}{\alpha^2},$$

and hence by (a),

$$A\left(x^{\gamma_1}\right) = \gamma_1\left(r + \frac{\alpha^2}{2}\left(\gamma_1 - 1\right)\right)x^{\gamma_1} = 0.$$

Note that since $X_t \to 0$ a.s. (a fact from the problem text), it follows that $\tau_{\rho,R} < \infty$ a.s. Let $\tau_k = \min(k, \tau_{\rho,R})$ (so that $E(\tau_k) \le k < \infty$), $k \in \mathbb{N}$, then by Dynkin,

$$\mathsf{E}^x\left(f\left(X_{\tau_k}\right)\right) \underset{Dynkin}{=} x^{\gamma_1} + 0,$$

where f is defined as above.

Let p_ρ be the probability that X_t hits the level R before the level ρ. Since $\tau_k \to \tau_{\rho,R}$ as $k \to \infty$ for all ω and f is bounded, we may use the dominated convergence theorem to show that

$$p_\rho R^{\gamma_1} + \left(1 - p_\rho\right)\rho^{\gamma_1} = E(f(X_{\tau_{\rho,R}})) = \lim_{k\to\infty} E(f(X_{\tau_k})) = x^{\gamma_1},$$

or

$$p_\rho = \frac{x^{\gamma_1} - \rho^{\gamma_1}}{R^{\gamma_1} - \rho^{\gamma_1}}. \tag{2}$$

We then send $\rho \to 0$ and find that $p_\rho \to p$, the probability of hitting level R before 0, $(1-p_\rho)\,\rho^{\gamma_1} \to 0$, and (hence)

$$p = \left(\frac{x}{R}\right)^{\gamma_1}.$$

Remark: An alternative proof is given in an appendix after part c). Here the k-approximation is avoided at the cost of a long direct computation to show that $E(\tau_{\rho,R}) < \infty$.

c) Since $X_t \to \infty$ a.s. it follows that $\tau < \infty$ a.s. Let $\tau_{\rho,R}$ and τ_k be as in part b), and take $f \in C_0^2$ such that

$$f(x) = \ln x \qquad \text{in} \qquad [\rho, R].$$

Here we have introduced $\tau_{\rho,R}$ to avoid $x = 0$ where the log is not continuous which precludes the use of Dynkin's formula.
Now since

$$A(\ln x) = rx\frac{d\ln x}{dx} + \frac{\alpha^2 x^2}{2}\frac{d^2\ln x}{dx^2} = r - \frac{\alpha^2}{2}\ (>0),$$

we use Dynkin's formula and find that

$$\begin{aligned}\mathsf{E}\left(f\left(X_{\tau_k}^x\right)\right) &= \ln x + \mathsf{E}\left(\int_0^{\tau_k}\left(r - \frac{\alpha^2}{2}\right)dt\right)\\ &= \ln x + \left(r - \frac{\alpha^2}{2}\right)\mathsf{E}\tau_k.\end{aligned}$$

We let p_ρ be as in b), the probability that the process hits $x = R$ before $x = \rho$, and send $k \to \infty$ using the dominated convergence theorem (f is bounded):

$$\begin{aligned}(1-p_\rho)\ln\rho + p_\rho \ln R &= \mathsf{E}\left(f\left(X_{\tau_{\rho,R}}^x\right)\right)\\ &= \lim_{k\to\infty}\mathsf{E}\left(f\left(X_{\tau_k}^x\right)\right) = \ln x + \left(r - \frac{\alpha^2}{2}\right)\mathsf{E}\tau_{\rho,R},\end{aligned}$$

or

$$\mathsf{E}\tau_{\rho,R} = \frac{(1-p_\rho)\ln\rho + p_\rho\ln R - \ln x}{r - \frac{\alpha^2}{2}}$$

Note that now $\gamma_1 < 0$ by assumption, so by (2) in part b),

$$p_\rho = \frac{x^{\gamma_1} - \rho^{\gamma_1}}{R^{\gamma_1} - \rho^{\gamma_1}} \to 1 \qquad \text{and} \qquad (1-p_\rho)\ln\rho \to 0$$

as $\rho \to 0$. Since $\tau_{\rho,R}$ is an increasing sequence converging to τ as $\rho \to 0$, we can use the monotone convergence theorem to conclude that

$$E(\tau) = \lim_{\rho\to 0} E(\tau_{\rho,R}) = \frac{\ln R - \ln x}{r - \frac{\alpha^2}{2}}.$$

The proof is complete.

Appendix: Alternative solution of b). We claim that $\mathsf{E}(\tau_{\rho,R}) < \infty$, and apply Dynkin's Formula with this stopping time:

$$p_\rho R^{\gamma_1} + (1 - p_\rho)\,\rho^{\gamma_1} = \mathsf{E}^x\left(f\left(X_{\tau_{\rho,R}}\right)\right) \underset{Dynkin}{=} x^{\gamma_1} + 0.$$

We let $\rho \to 0$ and find that $p_\rho \to p$, the probability of hitting level R before 0, $(1 - p_\rho)\,\rho^{\gamma_1} \to 0$, and

$$p = \left(\frac{x}{R}\right)^{\gamma_1}.$$

Let us prove the claim above. Note first that geometrical Brownian motion (Example 5.1.1) is given by

$$X_t = x \exp\left[\left(r - \frac{\alpha^2}{2}\right) t + \alpha B_t\right].$$

Remember that $r - \frac{\alpha^2}{2} < 0$, and let τ_ρ be the first time X_t hits the level ρ. We will show that $\mathsf{E}(\tau_\rho) < \infty$. Then we are done since $\tau_{\rho,R} \le \tau_\rho$ and hence

$$E(\tau_{\rho,R}) \le E(\tau_\rho) < \infty.$$

Consider

$$\begin{aligned} P\left(\omega\ ; \tau_\rho \ge t_0\ \right) &\le P\left(X_{t_0} \ge \rho\right) \\ &= P\left(\left(r - \frac{\alpha^2}{2}\right) t_0 + \alpha B_{t_0} \ge \log\left(\rho/x\right)\right) \\ &= P\left(B_{t_0} \ge \frac{\log\left(\rho/x\right) - \left(r - \frac{\alpha^2}{2}\right) t_0}{\alpha}\right). \end{aligned}$$

By Lemma 2 in the note on Brownian motion, *if X is $\mathcal{N}(0,1)$, then*

$$P\left(X \ge x\right) = 1 - \Phi\left(x\right) \le \sqrt{\frac{1}{2\pi}}\frac{1}{x}e^{-x^2/2}.$$

Hence,

$$\begin{aligned} P\left(B_{t_0} \ge A + Ct_0\right) &= 1 - \Phi\left(\frac{A + Ct_0}{t_0^{1/2}}\right) \\ &\le \frac{1}{\sqrt{2\pi}}\frac{t_0^{1/2}}{A + Ct_0}\exp\left[-\frac{1}{2}\left(\frac{A + Ct_0}{t_0^{1/2}}\right)^2\right], \end{aligned}$$

and

$$E(\tau_\rho) \le \sum_{k=1}^{\infty} kP(k - 1 \le \tau_\rho \le k) \le \sum_{k=1}^{\infty} kP(k - 1 \le \tau_\rho) < \infty.$$

NB! The same conclusion holds for a level *above* x in the case where $r - \frac{\alpha^2}{2} > 0$.

6 We consider Brownian motion in $\mathbb{R}^n$.

These exercises are best solved by making simple sketches.

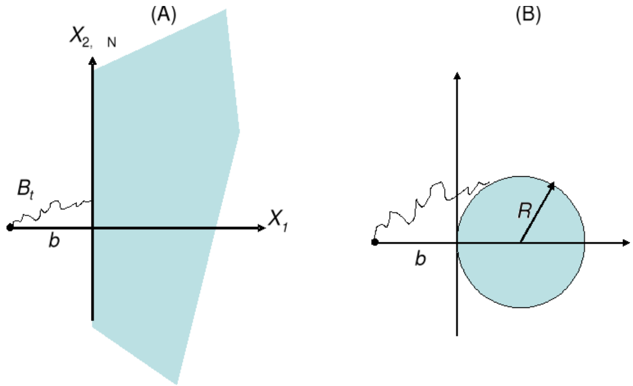

Figure 1: Two ways of proving that the probability of hitting a halfplane is always 1: (A) is to use the corresponding result for 1D Brownian motion. (B) is to use the result for the spere and letting $R \to \infty$ while b is constant.

a) There are several ways of seeing this, two methods are indicated in Fig. 1.

In (A) we have put the x-axis through the starting point ($\mathbf{x} = 0$) and orthogonal to (hyper)plane $\{x_1 \geq b\}, b > 0$. Hitting the half-space is the same as hitting $x_1 = b$ for the first component of the B.M. We know from 1d B.M. that the probability of hitting is 1, but the expected hitting time i ∞. That is, $\mathsf{E}^b(\tau_H) = \infty$.

For (B), the probability of hitting the half space is clearly larger than hitting the a smaller sphere within the half space. For $n = 1$ and $n = 2$, the probability of hitting any sphere is 1. For $n \geq 3$, we already know that

$$\mathsf{P}(\text{Hitting half-space}) \geq \mathsf{P}(\text{Hitting sphere}) = \frac{R^{n-2}}{(R+b)^{n-2}} \underset{R\to\infty}{\longrightarrow} 1.$$

Apparently not so easy to see that $\mathsf{E}^b(\tau_H) = \infty$ in this argument (?).

b) The simplest example is probably the 4d cylinder

$$C = \left\{(x_1, x_2, x_3, x_4)\ ;\ x_1^2 + x_2^2 + x_3^2 \leq 1,\ x_4 \in \mathbb{R}\right\}.$$

The probability of the 4d B.M. of hitting C from the outside is equal to the probability of the 3d B.M. of hitting the unit ball from the outside.

Let $U = \cup_{k\in\mathbb{N}} B()$ An example for $\mathbb{R}^3$ can be found is left open for discussion!

c) Many constructions should be possible. One idea is to think of a stack of cubed boxes set side by side so that starting inside any box, $\mathsf{E}^x(\tau_U) \leq 1$. All of $\mathbb{R}^3$ can be covered by such boxes and the walls of the boxes can be made gradually thinner (smaller volume) so that the total volume is finite. Take the set U to be the points belonging to the walls in this construction.

If a closed set U was sought, one could take the boundary of any sufficiently fine regular triangulation of $\mathbb{R}^n$.

7 The generator is

$$Au(x) = \alpha x \frac{\partial u}{\partial x} + \frac{1}{2}\beta^2 x^2 \frac{\partial^2 u}{\partial x^2}, \tag{3}$$

and the problem is of a form where the *Feynman-Kac* formula applies (*Theorem 8.2.1 in Øksendal*), except for the fact that $f(x)$ do not have compact support and is not differentiable at $x = K$.

Take a family $\{f_n(x)\}_{n=1}^{\infty}$ of non-negative $C_c^2(\mathbb{R})$-functions such that $\lim_{n\to\infty} f_n(x) = f(x)$. By the Feynman-Kac formula,

$$u_n(x,t) = E\left[e^{-\rho t} f_n(X_t^x)\right]. \tag{4}$$

satisfy

$$\frac{\partial u_n}{\partial t} = Au_n(x) - \rho u_n; \quad t > 0, \quad x \in \mathbb{R} \tag{5a}$$

$$u_n(0,x) = f_n(x); \quad x \in \mathbb{R}. \tag{5b}$$

The generator A is the generator of the geometrical Brownian motion

$$dX_t = \alpha X_t dt + \beta X_t dB_t \tag{6}$$

with solution

$$X_t = x \exp\left\{(\alpha - \frac{1}{2}\beta^2)t + \beta B_t\right\}. \tag{7}$$

Since the Brownian motion is Gaussian with zero mean and variance t, the solution to the p.d.e. can be written as

$$u_n(x,t) = \frac{e^{-\rho t}}{\sqrt{2\pi t}} \int_{\mathbb{R}} f_n(x \exp\{(\alpha - \frac{1}{2}\beta^2)t + \beta y\}) e^{-\frac{1}{2t}y^2} dy. \tag{8}$$

Taking the limit, and using the *dominated convergence theorem*,

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = \frac{e^{-\rho t}}{\sqrt{2\pi t}} \int_{\mathbb{R}} (x \exp\{(\alpha - \frac{1}{2}\beta^2)t + \beta y\} - K)^+ e^{-\frac{1}{2t}y^2} dy. \tag{9}$$

This is a candidate for a solution of the Black-Scholes equation. We must verify that it is the solution. This step is omitted, see the hints for how to do it.

Note: This formula can be simplified. Since the support of the integrand is

$$y > \frac{1}{2}\beta t - \frac{1}{\beta}(\ln \frac{x}{K} + \alpha t) := \gamma,$$

we may split the integral in two and complete the square to find that

$$\begin{aligned} u(x,t) &= \frac{e^{-\rho t}}{\sqrt{2\pi t}} \int_{\gamma}^{\infty} (x \exp\{(\alpha - \frac{1}{2}\beta^2)t + \beta y\} - K) e^{-\frac{1}{2t}y^2} dy \\ &= x e^{(\alpha-\rho)t} \Phi(\phi^+) - K e^{-\rho t} \Phi(\phi^-), \end{aligned}$$

where $\phi^{\pm} = \frac{1}{\beta\sqrt{t}}(\ln \frac{x}{K} + \alpha t) \pm \frac{1}{2}\beta\sqrt{t}$ and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy. \tag{10}$$

8 First we rewrite the stochastic differential equations in the vector form:

$$\begin{pmatrix} dS_t \\ dV_t \end{pmatrix} = \begin{pmatrix} \mu S_t \\ \alpha(\theta - V_t) \end{pmatrix} dt + \begin{pmatrix} \sqrt{V_t} S_t & 0 \\ 0 & \beta\sqrt{V_t} \end{pmatrix} \begin{pmatrix} dB_{1,t} \\ dB_{2,t} \end{pmatrix},$$

and note that the generator takes the form

$$A = \mu s \frac{\partial}{\partial s} + \alpha(\theta - v)\frac{\partial}{\partial v} + \frac{1}{2}\left(s^2 v \frac{\partial^2}{\partial s^2} + \beta^2 v \frac{\partial^2}{\partial v^2} \right). \tag{11}$$

Under the assumption that $f \in C_c^2(\mathbb{R})$ and that the interest rate $q(S_t, V_t) \equiv r$ is constant, we may apply the Feynmann-Kac formula and conclude that

$$w(x, v, t) = E\left(e^{-rt} f\left(S_t^{0,x,v} \right) \right)$$

satisfies the initial value problem

$$\begin{cases} \frac{\partial w}{\partial t} = Aw - rw, \quad t \in (0, T] \\ w(x, v, 0) = f(x). \end{cases} \tag{12}$$

Then we use the fact that the stochastic differential equation is time-homogeneous to find that the option price in the Heston model

$$\begin{aligned} u(x, v, t) &:= E\left(e^{-r(T-t)} f\left(S_T^{t,x,v} \right) \right) \\ &= E\left(e^{-r(T-t)} f\left(S_{T-t}^{0,x,v} \right) \right) \\ &= w(x, v, T - t). \end{aligned}$$

Then $\frac{\partial u}{\partial t} = -\frac{\partial w}{\partial t}$ and the derivatives of u and w with respect to s and v coinside. Hence by (12), it follows that

$$\begin{cases} \frac{\partial u}{\partial t} = -Au + ru, \quad t \in [0, T) \\ u\,(x, v, T) = f(x), \end{cases}$$

where the differential operator A is given in (11) above.