# 1 SOLUTION OF LINEAR STOCHASTIC EQUATIONS

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The linear stochastic equations make up an important class of models, and, similarly to the ordinary linear equations, there is a general analytic approach to their solution.

Recall that a regular first order differential equation of the form

$$\frac{dy(t)}{dt} = p(t)\,y + q(t) \tag{1}$$

may be solved by multiplying the equation by an integrating factor, $h(t)$,

$$hy' = h(py + q)\,,$$

so that

$$hy' = (hy)' - h'y = hpy + hq. \tag{2}$$

By choosing $h' = -hp$, that is, $h(t) = \exp\left(-\int_{t_0}^{t} p(s)\,ds\right)$, the innermost terms on both sides cancel, and we are left with

$$(hy)' = hq. \tag{3}$$

By integration, we obtain

$$h(t)\,y(t) - h(t_0)\,y(t_0) = \int_{t_0}^{t} h(s)\,q(s)\,ds, \tag{4}$$

or

$$y(t) = \frac{y(t_0) + \int_{t_0}^{t} h(s)\,q(s)\,ds}{h(t)}. \tag{5}$$

It turns out that the same trick also works for linear stochastic equations of the form

$$dX_t = p(t)\,X_t dt + q(t)\,dB_t. \tag{6}$$

Assume that $h(t)$ satisfies $h' = -hp$ as above, and consider the Itô process

$$Y_t = h(t)\,X_t. \tag{7}$$

If we apply the Itô formula, we obtain

$$\begin{aligned} dY_t &= h'(t)\,X_t dt + h(t)\,dX_t \\ &= h'(t)\,X_t dt + h(t)\left[p(t)\,X_t dt + q(t)\,dB_t\right] \\ &= h(t)\,q(t)\,dB_t. \end{aligned} \tag{8}$$

Now both sides may be integrated and

$$Y_t - Y_{t_0} = \int_{t_0}^{t} h(s)\,q(s)\,dB_s, \tag{9}$$

or

$$X_t = \frac{X_{t_0} + \int_{t_0}^{t} h(s)\, q(s)\, dB_s}{h(t)},$$
$$h(t) = \exp\left(-\int_{t_0}^{t} p(s)\, ds\right). \tag{10}$$

## 1.1 Physical Brownian Motion: The Ornstein-Uhlenbeck-Langevin Equation

Consider a small particle in a fluid, constantly and randomly hammered on by the surrounding fluid molecules. Although we are on the limits of continuous medium model when the diameter of the particle is $\mathcal{O}(10^{-6}\text{m})$, we may argue that since the dynamic viscosity of water or air is about $10^{-6}\text{m}^2/\text{s}$, a velocity of 1m/s would correspond to a Reynold's number of $\mathcal{O}(1)$, we are within the linear friction range. For one dimensional motion, the position $S(t)$ would then be determined by the (non-dimensional) equation of motion,

$$\frac{d^2 S(t)}{dt^2} + \mu \frac{dS(t)}{dt} = W(t). \tag{11}$$

Here $\mu$ is the viscous damping (or friction) coefficient and $W$ the "impulsive" excitation force. The impulsive force consists of random knocks varying in strength and moment of occurrence. The proper stochastic model for $W(t)$ is that of a Poisson point process.

We denote the particle velocity by $X$ and rewrite Eqn. 11 in our way as

$$dX_t = -\mu X_t dt + \sigma dB_t. \tag{12}$$

This is a quite famous model not only for physical Brownian motion (The Ornstein-Uhlenbeck model), but also for modelling so-called "shot noise" in electrical circuits (The Langevin model). The simple discrete variant of the model is

$$X_{j+1} - X_j = -\mu X_j \Delta t + \varepsilon_j, \tag{13}$$

or

$$X_{j+1} = (1 - \mu\Delta t) X_j + \varepsilon_j, \tag{14}$$

which we recognise as the standard auto-regressive stochastic process of order 1 (AR-1).

Eqn. 12 is a linear SDE of the form above with constant coefficients, and we leave to reader to show that the complete solution may be written as

$$X_t = X_0 e^{-\mu t} + \sigma \int_0^t e^{-\mu(t-s)} dBs. \tag{15}$$

The solution consists of a transient dying out as $e^{-\mu t}$, and a contribution witch keeps the motion going. The expectation of $X_t$ is simply

$$\mathsf{E}(X_t) = \mathsf{E}(X_0)\, e^{-\mu t}, \tag{16}$$

since the expectation of the Itô integral is always 0. For the variance of $X_t$, we make the reasonable assumption that $X_0$ is independent of the Brownian motion, and then, from Eqn. 15,

$$\mathrm{Var}\left(X_t\right) = e^{-2\mu t}\,\mathrm{Var}\left(X_0\right) + \mathrm{Var}\left(\sigma \int_0^t e^{-\mu(t-s)} dB_s\right). \tag{17}$$

By the Itô isometry,

$$\begin{aligned} \mathsf{E}\left(\sigma \int_0^t e^{-\mu(t-s)} dB_s\right)^2 &= \left\| \sigma \int_0^t e^{-\mu(t-s)} dB_s \right\|^2_{L^2(\Omega)} \\ &= \left\| \sigma e^{-\mu(t-s)} \right\|^2_{L^2([0,t]\times\Omega)} \qquad (18) \\ &= \sigma^2 \int_0^t e^{-2\mu(t-s)} ds \qquad (19) \\ &= \frac{\sigma^2}{2\mu}\left(1 - e^{-2\mu t}\right). \end{aligned}$$

From this we conclude that

$$\mathrm{Var}\left(X_t\right) = e^{-2\mu t}\,\mathrm{Var}\left(X_0\right) + \frac{\sigma^2}{2\mu}\left(1 - e^{-2\mu t}\right) \underset{t\to\infty}{\longrightarrow} \frac{\sigma^2}{2\mu}, \tag{20}$$

a reasonable result.

What about the position itself? Now

$$dS_t = X_t dt, \tag{21}$$

and hence

$$S_t = S_0 + \int_0^t X_s ds. \tag{22}$$

Since $X_t$ is clearly integrable on $L^2\left([0,t] \times \Omega\right)$, we may interchange $\int_0^t$ and $\mathsf{E}$, so that

$$\begin{aligned} \mathsf{E}S_t &= \mathsf{E}S_0 + \int_0^t \mathsf{E}\left(X_s\right) ds \\ &= \mathsf{E}S_0 + \int_0^t \mathsf{E}\left(X_0\right) e^{-\mu t} ds \qquad (23) \\ &= \mathsf{E}S_0 + \frac{\mathsf{E}X_0}{\mu}\left(1 - e^{-\mu t}\right) \end{aligned}$$

In the limit $t \to \infty$, we observe a permanent shift in the average position if $\mathsf{E}X_0 \neq 0$.

The variance of the particle is also interesting. For simplicity, we assume that $\mathsf{E}S_0 = 0$, $\mathsf{E}X_0 = 0$, so that

$$S_t = \int_0^t X_s ds = \int_0^t \left(\int_{u=0}^s \sigma e^{-\mu(s-u)} dB_u\right) ds. \tag{24}$$

This is an iterated integral, and we have not really discussed any type of Fubini-like theorems involving Itô integrals. However, in the present case, the integrand of the inner

Itô integral is so simple that it is no problems involved in interchanging the order of integration, hence

$$S_t = \sigma \int_0^t \left( \int_{s=u}^t e^{\mu(s-u)} ds \right) dB_u = \frac{\sigma}{\mu} \int_0^t \left(1 - e^{\mu(t-u)}\right) dB_u. \tag{25}$$

The integral then becomes a simple Itô integral, and by the Itô isometry,

$$\begin{aligned} \text{Var}\, S_t = \|S_t\|^2_{L^2([0\ t]\times\Omega)} &= \left(\frac{\sigma}{\mu}\right)^2 \int_0^t \left(1 - e^{\mu(t-s)}\right)^2 ds \\ &= \left(\frac{\sigma}{\mu}\right)^2 \left[ t + \frac{1}{2\mu}\left(-3 + 4e^{-\mu t} - e^{2\mu t}\right)\right] \qquad (26) \\ &\sim \begin{cases} \frac{\sigma^2}{3} t^3, & \text{for small } t\text{-s}, \\ \left(\frac{\sigma}{\mu}\right)^2 \left(t - \frac{3}{2\mu}\right), & \text{for large } t\text{-s}. \end{cases} \qquad (27) \end{aligned}$$

It is interesting to see that the behaviour for small values of $t$ is distinctly different from that of the standard Brownian motion.

## 1.2 Linear systems with stochastic excitation

Example 5.3.1. in BØ gives an example of a two-dimensional linear system with stochastic excitation. Such equations may be written in the general form

$$d\mathbf{X}_t = \left(\mathbf{A}\mathbf{X}_t + \mathbf{h}(t)\right) dt + \mathbf{K}(t)\, d\mathbf{B}_t, \tag{28}$$

where $\mathbf{X}_t, \mathbf{h}(t) \in \mathbb{R}^n$, $\mathbf{B}_t \in \mathbb{R}^m$, $\mathbf{A} \in \mathbb{R}^{n\times n}$, and $\mathbf{K}(t) \in \mathbb{R}^{n\times m}$. As usual, the integrating factor is the matrix exponential, $e^{-\mathbf{A}t}$. If we introduce

$$\mathbf{Y}_t = e^{-\mathbf{A}t}\mathbf{X}_t, \tag{29}$$

Itô's formula gives

$$d\mathbf{Y}_t = -\mathbf{A}e^{-\mathbf{A}t}\mathbf{X}_t dt + e^{-\mathbf{A}t} d\mathbf{X}_t, \tag{30}$$

and after multiplying Eqn. 28 by $e^{-\mathbf{A}t}$, we are left with

$$d\mathbf{Y}_t = e^{-\mathbf{A}t}\mathbf{h}(t)\, dt + e^{-\mathbf{A}t}\mathbf{K}(t)\, d\mathbf{B}_t, \tag{31}$$

from which it follows that

$$\mathbf{X}_t = e^{\mathbf{A}t}\mathbf{X}_{t_0} + \int_0^t e^{\mathbf{A}(t-s)}\mathbf{h}(s)\, ds + \int_0^t e^{\mathbf{A}(t-s)}\mathbf{K}(s)\, d\mathbf{B}_s. \tag{32}$$

## 1.3 Note

Some other equations which are solvable using an integrating factor is discussed in the exercises of Chapter 5 in BØ.