Martingales and the Itô Integral

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A *martingale* is a stochastic process modelling a fair betting game (the word "martingale" has at least three other meanings in English!). For a fair game, if X_n is the gambler's fortune after game number n, the expected change in the fortune after a later game m should be 0, regardless the information up to and including game n. This may be expressed in terms of the conditional expectation as

$$E\left(X_m - X_n|\mathcal{M}_n\right) = 0, \tag{1}$$

where $\mathcal{M}_n$ is the σ-algebra generated by events associated to the history of $\{X_i\}$, $i \leq n$. Since X_n itself is $\mathcal{M}_n$-measurable, $E\left(X_n|\mathcal{M}_n\right) = X_n$, and

$$E\left(X_m|\mathcal{M}_n\right) = X_n \tag{2}$$

for all $m \geq n$.

The formal definition of a martingale is given in BØ, starting with a *filtration* $\{\mathcal{M}_t\}$, that is, an increasing family of σ-algebras, $\mathcal{M}_s \subset \mathcal{M}_t \subset \mathcal{F}$, $s \leq t$.

The stochastic process $\{M_t\}$ *is a martingale w.r.t.* $\{\mathcal{M}_t\}$ *if*

(i) M_t is $\mathcal{M}_t$-measurable for all t-s,

(ii) $E\left|M_t\right| < \infty$ for all t-s,

(iii) $E\left(M_s|\mathcal{M}_t\right) = M_t$ for all $s \geq t$.

In the applications below, $\{M_t\}$ will be members of $L^2\left(\Omega, \mathcal{F}, P\right)$, and hence the conditional expectation w.r.t. $\mathcal{M}_t$ is the projection onto the closed subspace generated by all $\mathcal{M}_t$-measurable functions in $L^2\left(\Omega, \mathcal{F}, P\right)$ (See the first note, or read Lemma 6.1.1 in BØ). For a martingale, the projection of M_s for $s \geq t$ is simply M_t.

Below we shall give a slightly different derivation of Theorem 3.2.5 in BØ.

Let us for simplicity say that $\{M_t\}$ is an L^2*-martingale* if $E\left(\left|M_t\right|^2\right) < \infty$ (This implies that $E\left(\left|M_t\right|\right) < \infty$, since $P\left(\Omega\right)$ is finite).

Lemma 1: *Let* $\{M_t^n\}$, $n = 1, 2, \cdots$, *be a sequence of* L^2*-martingales w.r.t. to a common filtration* $\{\mathcal{M}_t\}$ *and assume that*

$$\left\|M_t^n - M_t\right\|_2^2 = E\left|M_t^n - M_t\right|^2 \underset{n\to\infty}{\longrightarrow} 0 \tag{3}$$

for all t*-s. Then* $\{M_t\}$ *is an* L^2*-martingale w.r.t.* $\{\mathcal{M}_t\}$.

Proof: All functions M_t^n are $\mathcal{M}_t$-measurable, and so is the limit function M_t (L^2-convergence in the closed subspace of $\mathcal{M}_t$-measurable functions). This proves (i), and since (ii) is obvious, only (iii) remains. However, the mapping $M \rightarrow E\left(M|\mathcal{M}_t\right)$ is a projection operator in $L^2\left(\Omega\right)$ and hence continuous, and since $\left\|M_s^n - M_s\right\|_2 \rightarrow 0$, $E\left(M_s^n|\mathcal{M}_t\right) = M_t^n$ converges *both* to $E\left(M_s|\mathcal{M}_t\right)$ and M_t.

Lemma 2: Let $\{M_t\}$ be an L^2*-martingale such that* $t \to M_t(\omega)$ *is continuous a.s. Then*

$$P\left(\sup_{0\le t\le T} |M_t| \ge \lambda\right) \le \frac{E\left(|M_T|^2\right)}{\lambda^2} \tag{4}$$

Proof: This is the L^2-case of *Doob's Martingale Inequality*. See references to its proof before BØ Theorem 3.2.4.

We now consider an Itô integral as a function of its upper limit,

$$M_t(\omega) = \int_0^t f(s,\omega)\, dB_s(\omega). \tag{5}$$

Theorem (BØ Theorem 3.2.5): *The Itô integral* $M_t(\omega) = \int_0^t f(t,\omega)\, dB_t(\omega)$ *is a martingale with respect to the filtration of the Brownian motion,* $\{\mathcal{F}_t\}$*. Moreover, the paths are t-continuous with probability 1.*

Proof: Let $\{\phi_n\}$ be a sequence of elementary functions converging to f in the definition of the Itô integral $\int_0^T f(s,\omega)\, dB_s(\omega)$. If we define

$$M_t^n(\omega) = \int_0^t \phi_n(s,\omega)\, dB_s(\omega), \tag{6}$$

it follows easily from the Itô isometry that $\|M_t^n - M_t\|_2 \underset{n\to\infty}{\longrightarrow} 0$ for *all* $t \in [0,T]$. Thus, by Lemma 1, $\{M_t\}$ is an L^2-martingale w.r.t. $\{\mathcal{F}_t\}$ if $\{M_t^n\}$ is. Now, from the definition of ϕ_n, M_t^n is clearly $\mathcal{F}_t$-measurable and in $L^2(\Omega,\mathcal{F},P)$. Assume that $0 \le t < s \le T$. Then, by the linearity of the conditional expectation and the Itô integral,

$$E\left(M_s^n|\mathcal{F}_t\right) = E\left(M_t^n|\mathcal{F}_t\right) + E\left(\int_t^s \phi_n(u)\, dB_u|\mathcal{F}_t\right) \tag{7}$$

$$= M_t^n + E\left(\int_t^s \phi_n(u)\, dB_u|\mathcal{F}_t\right). \tag{8}$$

It remains to be proved that the last term is equal to 0. The integral only consists of terms of the form

$$e_k\left(B_{t_{k+1}} - B_{t_k}\right), \tag{9}$$

where $t \le t_k < t_{k+1} \le s$, and where e_k is $\mathcal{F}_{t_k}$-measurable. Since $\mathcal{F}_t \subset \mathcal{F}_{t_k}$, we apply BØ, Theorem *B.2c*,*d*, and *e*, and *B.3* (check all steps!):

$$\begin{aligned} E\left[e_k\left(B_{t_{k+1}} - B_{t_k}\right)|\mathcal{F}_t\right] &= E\left[E\left(e_k\left(B_{t_{k+1}} - B_{t_k}\right)|\mathcal{F}_{t_k}\right)|\mathcal{F}_t\right] \\ &= E\left[e_k E\left(B_{t_{k+1}} - B_{t_k}|\mathcal{F}_{t_k}\right)|\mathcal{F}_t\right] \\ &= E\left[e_k E\left(B_{t_{k+1}} - B_{t_k}\right)|\mathcal{F}_t\right] \\ &= E\left[e_k \cdot 0|\mathcal{F}_t\right] = 0. \end{aligned} \tag{10}$$

Thus,

$$E\left(\int_t^s \phi_n(u)\, dB_u|\mathcal{F}_t\right) = 0. \tag{11}$$

Since Brownian motion has continuous paths, it is easy to see from the expression for the integral that also the paths of M_t^n are continuous a.s. (Check that the Itô integral of ϕ_n is continuous as a function of the upper limit t, – the function ϕ_n is not continuous!). In order to prove that the paths of the limit martingale M_t are continuous as well, the argument is somewhat similar to what was used for the continuity of Brownian motion. Since the difference between two martingales is also a martingale (check that), we first apply Lemma 2:

$$P\left(\sup_{0\leq t\leq T}|M_t^m - M_t^n| \geq \frac{1}{2^k}\right) \leq 2^{2k}\left\|M_t^m - M_t^n\right\|_2^2. \tag{12}$$

By choosing n_k sufficiently large, we have for all $m > n_k$ that $\left\|M_t^m - M_t^{n_k}\right\|_2^2 \leq 2^{-2k}2^{-k}$, and by repeating this argument, we obtain a subsequence of n-s, $\{n_k\}$ such that

$$P\left(\sup_{0\leq t\leq T}\left|M_t^{n_{k+1}} - M_t^{n_k}\right| \geq \frac{1}{2^k}\right) \leq \frac{1}{2^k},\ k = 1, 2, \cdots. \tag{13}$$

If we define

$$A_k = \left\{\omega;\ \sup_{0\leq t\leq T}\left|M_t^{n_{k+1}}(\omega) - M_t^{n_k}(\omega)\right| \geq \frac{1}{2^k}\right\}, \tag{14}$$

we then have

$$\sum_{k=1}^{\infty} P(A_k) < \infty. \tag{15}$$

By the easy part of the Borel-Cantelli Lemma, apart from a finite set of k-s,

$$\sup_{0\leq t\leq T}\left|M_t^{n_{k+1}}(\omega) - M_t^{n_k}(\omega)\right| < \frac{1}{2^k} \text{ a.s.} \tag{16}$$

Thus,

$$\sum_{k=1}^{\infty}\left(M_t^{n_{k+1}}(\omega) - M_t^{n_k}(\omega)\right) \tag{17}$$

is a "telescoping" series of continuous functions converging *uniformly* to the limit function $M_t(\omega)$ on $[0, T]$ with probability 1. This proves that $M_t(\omega)$ is indeed continuous on $[0\ T]$ with probability 1.