MA8109

Comments on Brownian Motion

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The Brownian motion will be the basic stochastic process we are considering in this course. Brownian motion is the mathematical idealization of a highly physical phenomenon, namely the irregular and unpredictable motion of microscopic particles in a fluid. The motion is due to the impact of bouncing molecules on the particle, and the phenomenon was discovered in 1828 by the Scottish botanist Robert Brown (Wikipedia reports about a brief controversy in the 1990-s dealing with whether Brown's microscope was good enough to see what he claimed to see). The mathematical model of Brownian motion was derived much later by Albert Einstein and Marian Smoluchowski.

Øksendal shows the existence of (mathematical) Brownian motion by means of two rather deep theorems by A. Kolmogorov, but it is, in fact, possible to derive Brownian motion by a much more constructive approach. This derivation also indicates how to make a computer programs able to simulate Brownian motion with any prescribed level of accuracy. We present the alternative approach based on an unpublished note by Professor Henrik H. Martens (1927–1993), and the argument is a nice application of the *Borel-Cantelli Lemma*. The construction goes apparently back to Paul-Pierre Lévy, one of the founders of modern probability theory.

1 Standard Brownian Motion

We start by postulating the properties we want Brownian motion to have (but can not yet say whether it really exists).

A one-dimensional standard Brownian motion (starting at $x = 0$ at $t = 0$*) is a stochastic process* $B(t)$*,* $t \geq 0$*, such that*

(i) $B(t)$ *is a Gaussian stochastic process,*

(ii) $E(B(t)) = 0$,

(iii) $E(B(s)B(t)) = \min(s,t)$.

(In the derivation below it is convenient to write $B(t)$ instead of using Øksendal's notation B_t).

Let us for the moment assume that it exists and consider some of its properties, following from the three postulates. First of all, the Brownian motion has *independent increments*. By this is understood that the change from time t_1 to t_2 is independent of the change from s_1 to s_2 if the intervals $[t_1, t_2]$ and $[s_1, s_2]$ are non-overlapping. This follows easily from (ii) and (iii) (assuming that $t_1 < t_2 \le s_1 < s_2$):

$$\begin{aligned}
&E\left[\left(B\left(t_2\right)-B\left(t_1\right)\right)\left(B\left(s_2\right)-B\left(s_1\right)\right)\right] \\
&=E\left[B\left(t_2\right)B\left(s_2\right)\right]-E\left[B\left(t_1\right)B\left(s_2\right)\right]-E\left[B\left(t_2\right)B\left(s_1\right)\right]+E\left[B\left(t_1\right)B\left(s_1\right)\right] \\
&=t_2-t_1-t_2+t_1=0.
\end{aligned} \tag{1}$$

The increments are thus *uncorrelated*, and hence *independent,* since all variables are (multivariate) Gaussian.

The variance of the increment increases linearly with time (assume $t_1 < t_2$):

$$\begin{aligned}
&\operatorname{Var}\left[\left(B\left(t_2\right)-B\left(t_1\right)\right)\right] \\
&=E\left[B\left(t_2\right)^2-2B\left(t_2\right)B\left(t_1\right)+B\left(t_1\right)^2\right] \\
&=t_2-2t_1+t_1=t_2-t_1.
\end{aligned} \tag{2}$$

Note that it is the variance and not the standard deviation that increases linearly with time.

The increments are *stationary* in the sense that

$$\begin{aligned}
&\operatorname{Var}\left[\left(B\left(t_2+s\right)-B\left(t_1+s\right)\right)\right] \\
&=t_2+s-t_1-s \\
&=t_2-t_1 \\
&=\operatorname{Var}\left[\left(B\left(t_2\right)-B\left(t_1\right)\right)\right].
\end{aligned} \tag{3}$$

Brownian motion is a *Markov process*: The future is only dependent on where we are, and *not* of the history up to the present position.

Before we continue, we observe that the problem of existence is related to the fact that time t is continuous. It is not difficult to produce a Brownian motion where t takes only discrete values. Let us first consider such a construction, and assume that time is discrete,

$$t=0,\Delta t,2\Delta t,\cdots,n\Delta t,\cdots. \tag{4}$$

Let $\Delta B_1,\cdots,\Delta B_n,\cdots$ be a sequence of independent, zero mean Gaussian variables with variance Δt, and define

$$\begin{aligned}
B_0&=0, \\
B_1&=\Delta B_1, \\
B_2&=B_1+\Delta B_2, \\
&\cdots \\
B_n&=B_{n-1}+\Delta B_n.
\end{aligned} \tag{5}$$

It is easy to see that the process $\{B_n\}_{n=0}^{\infty}$ satisfies the postulates (i) to (iii) above, including the last one, since

$$E\left(B_nB_m\right)=\min\left(n,m\right)\cdot\Delta t. \tag{6}$$

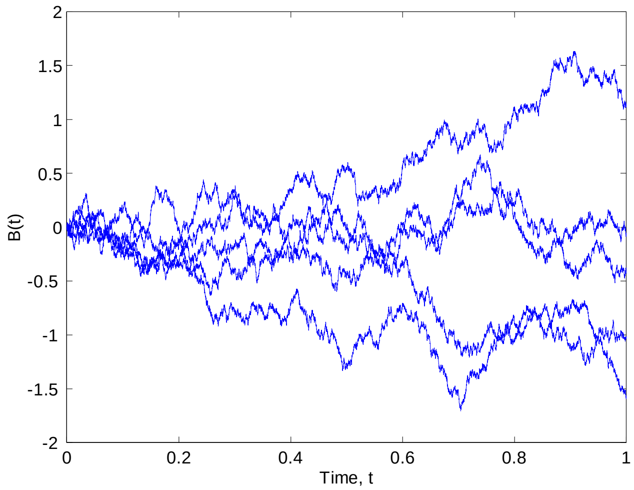

Figure 1: Five independent paths for a standard discrete Brownian motion, $\Delta t = 10^{-4}$.

By choosing Δt sufficiently small, we may thus obtain a discrete Brownian motion which mimics the continuous one (if it exists!) as close as we want.

Before you read further, you should start Matlab and try the following little program, which generates a discrete Brownian motion from $t = 0$ to $t = 1$:

```
N  = 10^5; Dt = 1/N;
DB = sqrt(Dt) * randn(N,1);
B  = cumsum(DB);
plot((1:N)*Dt,B);
```

Try to run a loop and accumulate several graphs on the same plot. Also try to experiment with the value of N. Five paths using this program are shown in Fig. 1.

The simulation shown in the figure represents an accurate and acceptable realization of a Brownian motion for many practical purposes. The simulation method is fast and reliable. The question is therefore why we now dig into the rather technical and difficult construction that eventually will lead to a Brownian motion defined for all t-s and with a.s. continuous paths. In a digital world like the present, there is no good answer to this question. The construction below is mathematically beautiful, and the existence of a continuous time Brownian motion leads in turn to the beautiful theory of the Itô integral, but is all the fuzz really necessary? With the tremendously powerful computers that everyone has access to, *all* practical work is digital, and all simulations *have to be discrete* by the very nature of the computer. It is therefore a good question whether it is at all necessary to bother with the continuous case. The answer will depend on your personal attitude, and it is acceptable to skip the following section and simply accept the conclusions ("*The mathematicians that insist that everything they use should be proved are welcome to apply this principle to their own PC-s*", citation of unknown origin).

2 Brownian Motion With Continuous Paths

We shall now construct a Brownian motion where the paths (realizations) are continuous for all t-s with probability 1. The argument is based on a construction that starts with the discrete Brownian motion above with $\Delta t = 1$. This defines the Brownian motion on the integers, $B(0)$, $B(1)$, $B(2)$, $\cdots$. Half-integer values are filled in by adding a random contribution to the linear interpolation, and this is repeated again and again. In the limit we obtain a function which we finally prove is the Brownian motion.

The argument depends on two lemmas which we state first:

Lemma 1: *Assume that the standard Brownian motion $B(t)$ exists and that $0 \leq t_1 < t_2$. Define*

$$Y = B\left(\frac{t_1 + t_2}{2}\right) - \frac{B(t_2) + B(t_1)}{2}. \tag{7}$$

Then $EY = 0$, $\operatorname{Var} Y = (t_2 - t_1)/4$, and Y is independent of $B(t_1)$ and $B(t_2)$.

Note that Y is the difference between the (hypothetical) Brownian motion at the midpoint between t_1 and t_2, and the line from $(t_1, B(t_1))$ to $(t_2, B(t_2))$.

Proof: It is obvious that $E(Y) = 0$, and the expression for $\operatorname{Var} Y$ follows from the postulates for the Brownian motion. In order to prove the independence of the endpoints, simply verify that

$$E(Y \cdot B(t_1)) = E(Y \cdot B(t_2)) = 0, \tag{8}$$

since we are dealing with Gaussian variables.

Lemma 2: *Let X be $\mathcal{N}(0,1)$. Then*

$$P(|X| > n) \leq \sqrt{\frac{2}{\pi}} \frac{1}{n} e^{-n^2/2}. \tag{9}$$

Proof: The proof consists of the following simple trick:

$$\begin{aligned} P(|X| > n) &= 2\int_n^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\ &= \sqrt{\frac{2}{\pi}} \int_n^\infty \frac{x}{x} e^{-x^2/2} dx \\ &\leq \sqrt{\frac{2}{\pi}} \frac{1}{n} \int_n^\infty x e^{-x^2/2} dx \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{n} e^{-n^2/2}. \end{aligned} \tag{10}$$

As we saw above, it is simple to define a discrete Brownian motion, so let us start with $\Delta t = 1$

and set

$$
\begin{aligned}
B(0) &= 0, \\
B(1) &= X_1, \\
B(2) &= B(1) + X_2, && (11) \\
&\cdots && (12) \\
B(n) &= B(n-1) + X_n, \\
&\cdots,
\end{aligned}
$$

where X_n are independent $\mathcal{N}(0,1)$ variables. By means of Lemma 1 this may be extended to a discrete Brownian motion on the half-integers by keeping $B(0)$, $B(1)$, $\cdots$, and defining, for each n,

$$B\left(n+\frac{1}{2}\right) = \frac{B(n) + B(n+1)}{2} + Y_n, \tag{13}$$

where Y is $\mathcal{N}(0, 1/4)$.

Let now $B_0(t)$ be the piecewise linear function passing through the points $B(0)$, $B(1)$, $\cdots$. This function, although it is defined for all t-s, is of course *not* a Brownian motion. We let $B_1(t)$ be the corresponding piecewise linear function passing through the integers and the half-integers. Note that B_1 may be written

$$B_1(t) = B_0(t) + \beta_1(t), \tag{14}$$

where $\beta_1(t)$ is a piecewise linear function that is 0 for $t = n$, and Y_n for $t = n + 1/2$ (make a drawing of B_0, B_1 and β_1!).

In general, we may define B_n as the piecewise linear function passing through all points of the form $t = k/2^n$, $k = 0, \cdots, n$ and write B_n in the form

$$B_n = B_0 + \beta_1 + \cdots + \beta_n. \tag{15}$$

Here $\{\beta_i\}$ are the extra contributions entering at the new mid-points at every step. Note that β_i is 0 at every second gridpoint, e.g. $\beta_2(t) = 0$ at $t = 0, 1/2, 1, \cdots$, non-zero at $t = 1/4, 3/4, \cdots$, and linear in between. The values at the non-zero gridpoints are all independent and Gaussian with 0 mean and a variance given by Lemma 1. In particular, the number of non-zero gridpoints in $[0,1]$ for β_n is 2^{n-1}, and the values at these gridpoints are independent zero mean Gaussian variables with variance equal to $1/2^{n+1}$ (check it!). The values may be generated from independent standard Gaussian variables X as $Y = 2^{-(n+1)/2} X$.

We are now going to prove that

$$\lim_{n\to\infty} B_n(t) = \lim_{n\to\infty} \left(B_0(t) + \beta_1(t) + \cdots + \beta_n(t)\right) = B(t), \tag{16}$$

where $B(t)$ is a continuous function (a.s.) fulfilling the postulates of the Brownian motion. The continuity of the limit function B will be guarantied if we are able to prove that the series converges absolutely and uniformly, that is,

$$\max_{0\leq t\leq 1}|B_0(t)|+\sum_{n=1}^{\infty}\max_{0\leq t\leq 1}|\beta_n(t)|<\infty. \tag{17}$$

(This is a general result from calculus). Because of the invariance of Brownian motion it is sufficient only to consider the time interval [0 1].

The following lemma provides the key we need.

Lemma 3: *The function family* $\{\beta_n\}$ *has the following property:*

$$P\left(\max_{0\leq t\leq 1}|\beta_n(t)|>\frac{n}{2^{(n+1)/2}}\text{ for infinitely many }n\text{-s}\right)=0. \tag{18}$$

(Note that in this lemma, we have a mapping from a probability space Ω which for a given ω provides all the Y-s we need for the construction of $\{\beta_n\}$. The lemma says that the stated condition only occurs for ω-s belonging to a set of measure 0).

Proof: The maximum of β_n has to be attained at one of the 2^{n-1} non-zero gridpoints with values $Y_i=2^{-(n+1)/2}X_i$, $i=1,\cdots,2^{n-1}$. The maximum is thus equal to

$$\frac{1}{2^{(n+1)/2}}\max\left(|X_1|,|X_2|,\cdots,X_{2^{n-1}}\right). \tag{19}$$

Hence,

$$\begin{aligned}P\left(\max_{0\leq t\leq 1}|\beta_n(t)|>\frac{n}{2^{(n+1)/2}}\right)&=P\left(\max_{k=1,\cdots,2^{n-1}}|X_k|>n\right)\\&\leq 2^{n-1}P\left(|X|>n\right)\\&\leq 2^{n-1}\sqrt{\frac{2}{\pi}}\frac{1}{n}e^{-n^2/2}.\end{aligned} \tag{20}$$

For the first inequality, observe that

$$\left\{\omega\ ;\max_{k=1,\cdots,2^{n-1}}|X_k(\omega)|>n\right\}\subset\bigcup_{k=1,\cdots,2^{n-1}}\left\{\omega\ ;|X_k(\omega)|>n\right\}. \tag{21}$$

The second inequality is Lemma 2.

If we now let

$$A_n=\left\{\omega\ ;\ \max_{0\leq t\leq 1}|\beta_n(t)|>\frac{n}{2^{(n+1)/2}}\right\}, \tag{22}$$

we obtain that

$$\sum_{n=1}^{\infty}P(A_n)\leq\sum_{n=1}^{\infty}2^{n-1}\sqrt{\frac{2}{\pi}}\frac{1}{n}e^{-n^2/2}<\infty \tag{23}$$

(Because the factor $e^{-n^2/2}$ kills everything in front). By the *Borel-Cantelli Lemma*, the set A_∞ of ω-s belonging to infinitely many A_n-s has probability 0. When $\omega\notin A_\infty$,

$$\max_{0\leq t\leq 1}|\beta_n(t)|\leq\frac{n}{2^{(n+1)/2}} \tag{24}$$

for all n-s apart from a finite number of them. This finite number has no influence on the convergence of the sequence in Eqn. 16, which, in fact, is then absolutely and uniformly convergent, since

$$\sum_{n=1}^{\infty} \frac{n}{2^{(n+1)/2}} < \infty. \tag{25}$$

There is nothing special with $t = 1$ in the above proof, so that the paths are continuous for all t-s with probability 1.

It remains to prove that the limit process $B(t)$ satisfies the postulates of Brownian motion. We first to observe that the vector $\mathbf{B}_n = \{B_n(t_1), B_n(t_2), \cdots, B_n(t_K)\}$ is multivariate Gaussian since each component is a linear sum of Gaussian variables, and this is also the case with any linear sum of its components (Øksendal, Thm. A.5). Since

$$\operatorname{Var}[B_n(t) - B_{n-1}(t)] = \operatorname{Var}\beta_n(t) \le \frac{1}{2^{n+1}}, \tag{26}$$

$\{B_n(t)\}$ will be a Cauchy sequence in the space $L^2(\Omega, P)$ and hence converge to a $B(t) \in L^2(\Omega, P)$. But this is the same limit function a.s., since we already know that $B_n(t)$ converges to $B(t)$ a.s. This applies to all components in $\mathbf{B}_n$, and the limit vector

$$\mathbf{B} = \{B(t_1), B(t_2), \cdots, B(t_K)\} \tag{27}$$

is Gaussian by Thm. A.7 in Øksendal. Thus, $B(t)$ is a Gaussian stochastic process. The rest of the postulates for Brownian motion follow by similar limit arguments.

In Øksendal, Brownian motion is constructed in $\mathbb{R}^n$ from the start. We leave to the reader to verify that the standard Brownian motion in $\mathbb{R}^n$ is just n independent components of the 1-D motion constructed above, that is, the *vector process*

$$\mathbf{B}(t) = \begin{pmatrix} B_1(t) \\ B_2(t) \\ \vdots \\ B_n(t) \end{pmatrix}. \tag{28}$$

An example of a 2-D discrete Brownian motion is displayed in Fig. 2.

The construction of the Brownian motion in Øksendal uses a rather deep theorem of A. Kolmogorov by first stating the joint Gaussian distributions for all finite collections $\{B(t_1), \cdots, B(t_K)\}$. It is easy to specify this distribution by first defining the probability density p of Y with mean x and covariance matrix tI,

$$p(t, x, y) = \frac{2}{(2\pi t)^{n/2}} \exp\left(-\frac{|y - x|^2}{2t}\right), \; x, y \in \mathbb{R}^n. \tag{29}$$

The joint density of $\{B(t_1), \cdots, B(t_K)\}$ (starting at 0) with $0 \le t_1 \le \cdots \le t_k$ is simply

$$p(t_1, 0, x_1)\, p(t_2 - t_1, x_1, x_2) \cdots p(t_K - t_{K-1}, x_{K-1}, x_K). \tag{30}$$

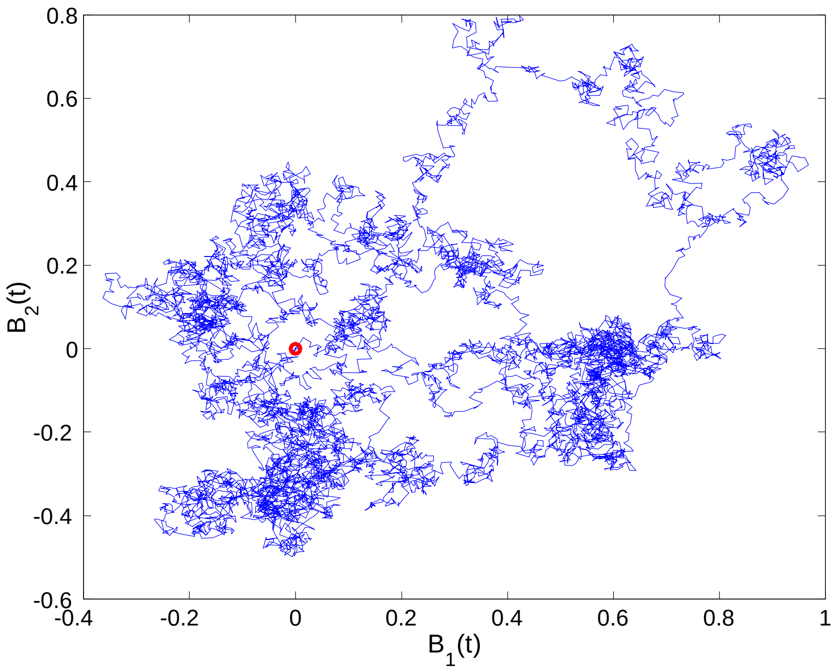

Figure 2: Simulated 2-D standard Brownian motion starting at $(0,0)$, $\Delta t = 10^{-4}$, $0 \leq t \leq 1$. Where is the end?

This is reasonable: $B(t_1)$ is $\mathcal{N}(0, t_1)$, and since $B(t_1)$ and $B(t_2) - B(t_1)$ are independent, $\{B(t_1), B(t_2)\}$ has the joint density

$$p(t_1, 0, x_1)\, p(t_2 - t_1, x_1, x_2)\,. \tag{31}$$

The corresponding measure may be extended to fulfill Kolmogorov's consistency properties, and therefore $B(t)$ exists. The continuity follows from another of Kolmogorov's theorems.

3 The Irregularity of Brownian Motion

Even if we have proved that the paths are continuous, the graphs indicate that Brownian motion does not consists of smooth and nice functions. In this section we shall prove that the paths are indeed quite irregular.

This section may be skipped completely without any significant loss of information.

A partition $\mathcal{P}$ of the interval $[a, b]$ is a set of points $\{t_k\}$ such that

$$a = t_1 < t_2 < \cdots < t_N = b. \tag{32}$$

The limit $\mathcal{P} \to 0$ means that $\max_k |t_k - t_{k-1}| = \max_k |\Delta t_k| \to 0$. The *p-variation* of a continuous function defined on $[a, b]$ is defined as

$$M_p = \overline{\lim_{\mathcal{P}\to 0}} \sum_{\mathcal{P}} |g(t_k) - g(t_{k-1})|^p \tag{33}$$

($\overline{\lim}$ means "limsup"). In particular, g is said to be of *bounded variation* if the 1-variation is finite. It is easy to prove (by the triangle inequality) that the limit M_1 in this case is actually

$$M_1 = \sup_{\mathcal{P}} \sum_{\mathcal{P}} |g(t_k) - g(t_{k-1})| \,. \tag{34}$$

Lemma 4: *A continuous function of bounded variation on a finite interval has 0 quadratic variation.*

Proof: Set $M_1 = \sup_{\mathcal{P}} \sum_{\mathcal{P}} |g(t_k) - g(t_{k-1})|$. Note that since the interval is finite, g is uniformly continuous, i.e. for any $\varepsilon > 0$ there is a $\delta(\varepsilon)$ such that

$$\max_k |g(t_k) - g(t_{k-1})| \leq \varepsilon \tag{35}$$

whenever $\max_k |t_k - t_{k-1}| \leq \delta(\varepsilon)$. Then

$$\begin{aligned} \sum_{\mathcal{P}} |g(t_k) - g(t_{k-1})|^2 &\leq \max_k |g(t_k) - g(t_{k-1})| \sum_{\mathcal{P}} |g(t_k) - g(t_{k-1})| \\ &\leq \varepsilon M_1. \end{aligned} \tag{36}$$

Since the bound εM_1 holds for *all* partitions where $\max_k |t_k - t_{k-1}| \leq \delta(\varepsilon)$, we have $M_2 \leq \varepsilon M_1$. Thus, M_2 has to be 0.

Lemma 5: *For a standard Brownian motion $B(t)$ defined on* $[0, T]$,

$$M_2 = \overline{\lim_{\mathcal{P}\to 0}} \sum_{\mathcal{P}} |B(t_k) - B(t_{k-1})|^2 \geq T \text{ a.s.} \tag{37}$$

Corollary: *The paths of a standard Brownian motion have infinite variation a.s.!*

Proof, Corollary: Since the quadratic variation is larger or equal to T a.s., the paths can not be of bounded variation according to Lemma 4.

Note that although

$$E\left(\sum_{\mathcal{P}} |B(t_k) - B(t_{k-1})|^2\right) = \sum_{\mathcal{P}} (t_k - t_{k-1}) = T, \tag{38}$$

The statement in Lemma 5 is *much* stronger.

Proof, Lemma 5: Let $Y_{\mathcal{P}}$ be the stochastic variable

$$Y_{\mathcal{P}} = \sum_{\mathcal{P}} |B(t_k) - B(t_{k-1})|^2 \tag{39}$$

We first want to prove that

$$\|Y_{\mathcal{P}} - T\|_2^2 = E\left(Y_{\mathcal{P}} - T\right)^2 \underset{\mathcal{P}\to 0}{\longrightarrow} 0. \tag{40}$$

We just observed that $EY_{\mathcal{P}} = T$. Moreover, with $\Delta B_k = B(t_k) - B(t_{k-1})$,

$$EY_{\mathcal{P}}^2 = E\left(\sum_k \Delta B_k^2 \sum_l \Delta B_l^2\right) = \sum_k \sum_l E\left(\Delta B_k^2 \Delta B_l^2\right). \tag{41}$$

Since we are dealing with Gaussian variables,

$$E\left(\Delta B_k^2 \Delta B_l^2\right) = \begin{cases} \Delta t_k \Delta t_l, & l \neq k \\ 3\Delta t_k^2, & l = k \end{cases}, \tag{42}$$

where we have applied the *four-cumulant identity* for zero mean Gaussian variables (See Exercise Set 1),

$$E\left(X_1X_2X_3X_4\right) = E\left(X_1X_2\right)E(X_3X_4) + E\left(X_1X_3\right)E(X_2X_4) + E\left(X_1X_4\right)E(X_2X_3)\,. \tag{43}$$

Thus,

$$EY_{\mathcal{P}}^2 = 2\sum_k \Delta t_k^2 + \sum_k \Delta t_k \sum_l \Delta t_l = 2\sum_k \Delta t_k^2 + T^2. \tag{44}$$

Putting this together,

$$\begin{aligned} E\left(Y_{\mathcal{P}} - T\right)^2 &= EY_{\mathcal{P}}^2 - 2TEY_{\mathcal{P}} + T^2 \\ &= 2\sum_k \Delta t_k^2 + T^2 - 2T^2 + T^2 \\ &= 2\sum_k \Delta t_k^2 \\ &\leq 2\max_k \left|t_k - t_{k-1}\right| T \underset{\mathcal{P}\to 0}{\longrightarrow} 0. \end{aligned} \tag{45}$$

Since mean square convergence implies pointwise convergence a.s. for a subset of partitions (L^p-theory), it is clear that the quadratic variation has to be larger or equal to T a.s.