Mean-Square Continuous Stochastic Processes

Harald E. Krogstad, 2011

This note should be read together with Problem 3.13 and the proof of Theorem 4.1.2 in BØ, and collects some results for this important class of stochastic processes. The term Mean-Square refers to the norm $\|\|_{L^2(\Omega,\mathcal{F},P)}$.

For mathematical models originating from real world applications, stochastic variables tend to have finite mean and variance, and stochastic processes tend to be *mean square continuous*, that is,

$$\|X_t - X_s\| \underset{t\to s}{\longrightarrow} 0. \tag{1}$$

Here and below, the norm is $\|\|_{L^2(\Omega,\mathcal{F},P)}$. Thus, if $t \to \mathsf{E}(X_t)$ is a continuous function,

$$\operatorname{Var}(X_t - X_s) \underset{t\to s}{\longrightarrow} 0 \tag{2}$$

(check!). This does not imply that the paths themselves are continuous and, conversely, having continuous paths does not necessarily imply that the stochastic process is mean square continuous.

An assumption about mean square continuity simplifies the arguments when we deal with integrals of stochastic functions.

First of all, if X_t is mean-square continuous, the function $t \to \|X_t\|$ will be continuous, as follows from the *inverted* triangular inequality,

$$|\|X_t\| - \|X_s\|| \le \|X_t - X_s\|. \tag{3}$$

(check!).

Brownian motion is mean square continuous since $\mathsf{E}(B_t)$ is constant, and $\operatorname{Var}(B_t - B_s) = |t-s|$. Any weakly stationary stochastic process X_t is mean-square continuous since $\mathsf{E}(X_t)$ is constant and the covariance function $\rho_X(t)$ is continuous:

$$\rho_X(t) - \rho_X(s) = \int_{\hat{\mathbb{R}}} \left(e^{i\nu t} - e^{i\nu s}\right) d\mu_X(\nu) \underset{s\to t}{\longrightarrow} 0 \tag{4}$$

by dominated convergence, since μ_X is a bounded measure on $\hat{\mathbb{R}}$. Hence,

$$\operatorname{Var}(X_t - X_s) = \operatorname{Var}(X_t) - 2\operatorname{Cov}(X_t, X_s) + \operatorname{Var}(X_s) = 2\rho_X(0) - 2\rho_X(t-s) \underset{s\to t}{\longrightarrow} 0. \tag{5}$$

Similar to ordinary continuous functions defined on a finite interval, a mean-square continuous random process on the finite interval $[S,T]$ will be *mean square uniformly continuous*:

Lemma 1: *If $X(t)$ is a mean-square continuous process on the finite interval$[S,T]$, then for all fixed $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ so that*

$$\|X_t - X_s\| < \varepsilon, \tag{6}$$

whenever $|t-s| < \delta(\varepsilon)$.

Proof: The proof is similar to the proofs for ordinary continuous functions, using compactness. The simplest is probably to assume there is an ε where no such $\delta > 0$ works. Then, for each $n \geq 1$, there are $s_n, t_n \in [S, T]$ so that $\|X_{t_n} - X_{s_n}\| > \varepsilon$, whereas $|s_n - t_n| < \frac{1}{n}$. By the compactness of the closed, finite interval $[S, T]$, pick a subsequence from $\{s_n\}$ convergent to $s_0 \in [S, T]$. Because $|s_n - t_n| < \frac{1}{n}$, the corresponding t-subsequence is convergent to the same limit. This contradicts the assumption that $X(t)$ is mean square continuous at s_0.

For a stochastic process f_t,

$$R(\omega) = \int_S^T f_t(\omega)\, dt \tag{7}$$

is just the regular integral of the realization $f_t(\omega)$ (The existence of such integrals, with probability 1, is assumed to be part of the definition of f_t).

Lemma 2: *If f_t is a stochastic process on $[S, T]$ with $f_t \in L^2(\Omega, \mathcal{F}, P)$, then*

$$\|R\| \leq \int_S^T \|f_t\|\, dt. \tag{8}$$

Proof: The proof requires *Schwarz' Inequality* and *Tonelli's Theorem*:

$$\begin{aligned}
\|R\|^2 &= \int_\Omega \left(\int_S^T f_t(\omega)\, dt\right)\left(\int_S^T f_s(\omega)\, ds\right) dP(\omega) \\
&\leq \int_\Omega \left(\int_S^T |f_t(\omega)|\, dt\right)\left(\int_S^T |f_s(\omega)|\, ds\right) dP(\omega) \\
&= \int_S^T \int_S^T \left(\int_\Omega |f_t(\omega)|\, |f_s(\omega)|\, dP(\omega)\right) dt ds \\
&\leq \int_S^T \int_S^T \|f_t\|\, \|f_s\|\, dt ds \\
&= \left(\int_S^T \|f_t\|\, dt\right)^2.
\end{aligned} \tag{9}$$

Recall the definition of a partition $\mathcal{P}$ of the interval $[S, T]$, and what it means that $\mathcal{P} \to 0$.

Proposition 1: *If f_t is a mean square continuous stochastic process on $[S, T]$, then*

$$\int_S^T f_t dt = \lim_{\mathcal{P} \to 0} \sum_{\mathcal{P}} f_{t_j}(t_{j+1} - t_j) \tag{10}$$

Proof: Write

$$\sum_{\mathcal{P}} f_j \Delta t_j - \int_S^T f_t dt = \sum_{\mathcal{P}} \int_{t_j}^{t_{j+1}} \left(f_{t_j} - f_t\right) dt, \tag{11}$$

and apply Lemma 1 and Lemma 2.

Similar to the regular Stieltjes integral, the integral of mean square continuous functions exists whenever the distribution function $\alpha(t)$ has bounded variation:

$$\int_S^T f_t d\alpha(t) = \lim_{\mathcal{P}\to 0} \sum_j f_{\tau_j} \left[\alpha(t_{j+1}) - \alpha(t_j)\right], \ \tau_j \in [t_j, t_{j+1}). \tag{12}$$

However, this can not be immediately applied for the Itô integral, since Brownian motion paths have infinite variation with probability 1. Nevertheless, it turns out that the Itô integral is what we expect for mean-square continuous functions.

Proposition 2 (BØ, Problem 3.13): *If* f_t *is a mean square continuous* $\mathcal{F}_t$*-adapted function on the finite interval* $[S,T]$*, the Itô integral of* f_t *may be computed by the formula*

$$\int_S^T f_t dB_t = \lim_{\mathcal{P}\to 0} \int_S^T \phi_t^{(n)} dB_t \tag{13}$$

where

$$\phi_t^{(n)} = \sum_{\mathcal{P}} f_{t_j} \chi_{[t_j, t_{j+1})}, \tag{14}$$

i.e. by using the function value at the left endpoint of the intervals in the partition.

Proof: The functions $\phi_t^{(n)}$ are clearly $\mathcal{F}_t$-adapted, and, for any $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ for f_t as in Lemma 1 above. Thus, for any partition $\mathcal{P}$ where $\max_j |t_{j+1} - t_j| < \delta(\varepsilon)$, we have

$$\begin{aligned} \left\| f_t - \phi_t^{(n)} \right\|^2_{L^2([S,T]\times\Omega)} &= \mathsf{E}\left(\sum_j \int_{t_j}^{t_{j+1}} \left| f_t - f_{t_j} \right|^2 dt \right) \\ &\le \sum_j \varepsilon (t_{j+1} - t_j) = \varepsilon (T - S), \end{aligned} \tag{15}$$

and the limit for the integrals in Eq. 13 follows from the Itô Isometry.

Mean square continuity makes also the main arguments in the derivation of *Itô's Formula* quite transparent, as stated in the next lemma.

Lemma 3: *Let* $[S,T]$ *be a finite interval and* f_t *a mean square continuous,* $\mathcal{F}_t$*-adapted function. Then (with* $f_j = f_{t_j}$*,* $\Delta t_j = t_{j+1} - t_j$*, and* $\Delta B_j = B_{t_{j+1}} - B_{t_j}$*), the following* $L^2(\Omega)$*-limits apply*:

$$\begin{array}{cc} (i) & \sum_{\mathcal{P}} f_j (\Delta t_j)^2 \underset{\mathcal{P}\to 0}{\longrightarrow} 0, \\ (ii) & \sum_{\mathcal{P}} f_j \Delta t_j \Delta B_j \underset{\mathcal{P}\to 0}{\longrightarrow} 0, \\ (iii) & \sum_{\mathcal{P}} f_j (\Delta B_j)^2 \underset{\mathcal{P}\to 0}{\longrightarrow} \int_S^T f_t dt, \\ (iv) & \sum_{\mathcal{P}} f_j (\Delta t_j)^n (\Delta B_j)^m \underset{\mathcal{P}\to 0}{\longrightarrow} 0 \text{ when } n + m \ge 3. \end{array} \tag{16}$$

Proof: Since $[S,T]$ is bounded, $\|f_t\| \le M < \infty$. Then (i) is trivial since

$$\left\| \sum_{\mathcal{P}} f_j (\Delta t_j)^2 \right\| \le \max_j \Delta t_j \cdot M \cdot \sum_{\mathcal{P}} \Delta t_j = M (T - S) \max_j \Delta t_j. \tag{17}$$

For (ii), we apply that f_t is $\mathcal{F}_t$-adapted to eliminate all terms of the form $\mathsf{E}\left(f_i f_j \Delta B_i \Delta B_j\right)$ when $i \neq j$, and hence,

$$\begin{aligned}\left\|\sum_{\mathcal{P}} f_j \Delta t_j \Delta B_j\right\|^2 &= \sum_{\mathcal{P}} \mathsf{E}\left(f_j\right)^2 \mathsf{E}\left(\Delta B_j\right)^2 \left(\Delta t_j\right)^2 = \\ &= \sum_{\mathcal{P}} \mathsf{E}\left(f_j\right)^2 \left(\Delta t_j\right)^3 \leq \max_j \left(\Delta t_j\right)^2 M^2 \left(T-S\right) \underset{\mathcal{P}\to 0}{\longrightarrow} 0. \end{aligned} \tag{18}$$

For (iii) we first consider

$$\begin{aligned}\left\|\sum_{\mathcal{P}} f_j \left(\Delta B_j\right)^2 - \sum_{\mathcal{P}} f_j \Delta t_j\right\|^2 &= \left\|\sum_{\mathcal{P}} f_j \left(\left(\Delta B_j\right)^2 - \Delta t_j\right)\right\|^2 \\ &= \mathsf{E}\left(\sum_{\mathcal{P},\mathcal{P}} f_i f_j \left(\left(\Delta B_i\right)^2 - \Delta t_i\right)\left(\left(\Delta B_j\right)^2 - \Delta t_j\right)\right) \\ &\overset{(*)}{=} \mathsf{E}\left(\sum_{\mathcal{P}} f_j^2 \left(\left(\Delta B_j\right)^2 - \Delta t_j\right)^2\right) \\ &= \sum_{\mathcal{P}} \mathsf{E}\left(f_j^2\right) \mathsf{E}\left(\left(\Delta B_j\right)^4 - 2\left(\Delta B_j\right)^2 \Delta t_j + \left(\Delta t_j\right)^2\right) \\ &= \sum_{\mathcal{P}} \mathsf{E}\left(f_j^2\right)\left(3\left(\Delta t_j\right)^2 - 2\left(\Delta t_j\right)^2 + \left(\Delta t_j\right)^2\right) \\ &\leq 2M^2 \sum_{\mathcal{P}} \left(\Delta t_j\right)^2 \underset{\mathcal{P}\to 0}{\longrightarrow} 0. \end{aligned} \tag{19}$$

For the $(*)$-step, note that $f_i f_j \left(\left(\Delta B_i\right)^2 - \Delta t_i\right)$ is independent of $\left(\left(\Delta B_j\right)^2 - \Delta t_j\right)$ for $i < j$, and *vice versa*. The conclusion then follows from Proposition 1.

Statement (iv) is left for the reader.

The following (sketch of a) proof of *Itô's Formula* assumes that all processes are mean-square continuous.

We start with a mean-square continuous Itô Process X_t with representation

$$dX_t = u_t dt + v_t dB_t, \tag{20}$$

where u_t and v_t are adapted mean-square continuous processes. The process X_t is transformed by

$$X_t \to Y_t = g\left(t, X_t\right), \tag{21}$$

where g is smooth enough for the transformation and the derivations below. Introduce a partition $\mathcal{P}$ of the interval $[0,T]$ and write

$$Y_T - Y_0 = g\left(T, X_T\right) - g\left(0, X_0\right) = \sum_{\mathcal{P}} g\left(t_{j+1}, X_{t_{j+1}}\right) - g\left(t_j, X_{t_j}\right) = \sum_{\mathcal{P}} \Delta g_j. \tag{22}$$

The task is now to prove that the sum converges to the integral representation for Y_t when $\mathcal{P} \to 0$. Expanding Δg_j in a Taylor series leads to

$$\Delta g_j = g_{xj}\Delta X_j + g_{tj}\Delta t + \frac{1}{2}\left\{g_{xxj}\Delta X_j^2 + 2g_{xtj}\Delta X_j\Delta t + g_{ttj}\Delta t^2\right\} + \cdots , \tag{23}$$

where $g_{xtj} = \frac{\partial^2 g}{\partial x \partial t}\left(t_j, X_{t_j}\right)$ etc., and

$$\Delta X_j = \int_{t_j}^{t_{j+1}} \left\{u_t dt + v_t dB_t\right\}. \tag{24}$$

However, the leading order approximation to ΔX_j is $u_j\Delta t_j + v_j\Delta B_j$, and writing $\Delta X_j = u_j\Delta t_j + v_j\Delta B_j + e_j$ (e_j is the error), we obtain

$$\begin{aligned}
\|e_j\| &= \|\Delta X_j - (u_j\Delta t_j + v_j\Delta B_j)\| \\
&= \left\| \int_{t_j}^{t_{j+1}} (u_t - u_j)\, dt + \int_{t_j}^{t_{j+1}} (v_t - v_j)\, dB_t \right\| \\
&\leq \left\| \int_{t_j}^{t_{j+1}} (u_t - u_j)\, dt \right\| + \left\| \int_{t_j}^{t_{j+1}} (v_t - v_j)\, dB_t \right\| \\
&\leq \varepsilon \Delta t_j + \left(\int_{t_j}^{t_{j+1}} \|v_t - v_j\|^2\, dt \right)^{1/2},
\end{aligned} \tag{25}$$

by Lemma 2 and the Itô Isometry, also assuming that the partition is fine enough to ensure $\|u - u_j\| \leq \varepsilon$. For the last term in the RHS of Eq. 25, we shall assume that v_t is L^2-Lipschitz continuous, that is $\|v_t - v_s\| \leq M\,|t - s|$ (for a non-random v_t, this is the regular Lipschitz condition). Then

$$\int_{t_j}^{t_{j+1}} \|v_t - v_j\|^2\, dt \leq M^2 \int_0^{\Delta t_j} t^2 dt = \frac{M^2}{3}\Delta t_j^3, \tag{26}$$

and

$$\|e_j\| \leq \left(\varepsilon + \left(\frac{M^2}{3}\Delta t_j \right)^{1/2} \right) \Delta t_j. \tag{27}$$

We substitute ΔX_j by $u_j\Delta t_j + v_j\Delta B_j + e_j$ in Eq. 23 and 22. If $\mathcal{P} \to 0$ in Eq. 22, the contribution from the error terms vanishes, and we obtain from Proposition 1, 2, and Lemma 3:

$$Y_T - Y_0 = \int_0^T \left\{ \frac{\partial g}{\partial t} + u\frac{\partial g}{\partial x} + \frac{v^2}{2}\frac{\partial^2 g}{\partial x^2} \right\} dt + \int_0^T v dB_t, \tag{28}$$

and the differential form

$$dY_t = \left\{ \frac{\partial g}{\partial t}(t, X_t) + u_t\frac{\partial g}{\partial x}(t, X_t) + \frac{v_t^2}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t) \right\} dt + v_t dB_t. \tag{29}$$