SDE 17.10.2019

Info: - Projects

- Ref grp. meeting tomorrow

Chp. 5: SDE's

5.1 Definitions

DE driven by (Gaussian) white noise

$$\frac{dX_t}{dt} = b(t,X_t) + \sigma(t,X_t)\, W_t$$

Ito's interpretation (int. form):

(1) $$\underset{\mathbb{R}^n}{\underset{\Uparrow}{X_t}} = X_0 + \int_0^t \underset{\mathbb{R}^n}{\underset{\Uparrow}{b}}(s,X_s)\,ds + \int_0^t \underset{\mathbb{R}^{n\times m}}{\underset{\Uparrow}{\sigma}}(s,X_s)\,d\underset{\mathbb{R}^m}{\underset{\Uparrow}{B}}_s$$

$\Updownarrow$ notation

$$dX_t = \underbrace{b(t,X_t)\,dt}_{\text{drift}} + \underbrace{\sigma(t,X_t)\,dB_t}_{\text{exitation}}$$

σ = volatility (finance)

X_t should be Ito-proc.:

filtered prob. sp.

Def. 1: $(\Omega, \mathcal{F}, \{\mathcal{H}_t\}_t, P)$, B_t $\mathbb{R}^m$ cannonical $\{\mathcal{H}_t\}_t$ - B.M.

X_t strong sol'n of (1) for $t \in [0,\infty)$ if

i) X_t is $\{\mathcal{H}_t\}_t$ - adap. and $\mathcal{B}_{[0,\infty)} \times \mathcal{F}$ - meas.

ii) $b(t,X_t), \sigma(t,X_t)$ ——— " ———

$$\int_0^t (|b(s,X_s)| + |\sigma(s,X_s)|^2)\, ds < \infty \quad \forall t > 0 \text{ a.s.}$$

iii) Eq'n (1) holds ~~$\forall t \geq 0$ a.s.~~

a.s. for all t >= 0

Obs. 2:

a) $b(t,X_t) \in \tilde{\mathcal{L}}^1_{(0,t), \mathcal{H}_t}$, $\sigma(t,X_t) \in \tilde{\mathcal{L}}^2_{(0,t), \mathcal{H}_t}$!

b) X_t Ito-proc. by i), ii) $(\Rightarrow t\text{-cont. a.s.})$

c) b, σ cont.; X_t cont. a.s. + i) $\Rightarrow$ ii) holds

Analytic sol'ns:

Ito-formula + ODE-meth's and estim's (HW!)
(see HW prob'ms!)

5.2 Ex: Population growth

Std. model: $dN_t = r N_t dt \Rightarrow N_t = N_0 e^{rt}$

Rnd. growth rate:

$$r\,dt \longrightarrow r\,dt + \alpha\, dB_t \quad (\text{Ito})$$

$$(2) \quad dN_t = r N_t dt + \alpha N_t dB_t \quad (u = rN,\ v = \alpha N)$$

$$\int_0^t \frac{dN_s}{N_s} = rt + \alpha B_t$$

$$\overset{\textcircled{2}}{=} \quad \longleftrightarrow \quad \int d \ln N_t$$

Use Ito formula!

$$Y_t = \ln N_t, \quad g(t,x) = \ln x, \quad g_x = \frac{1}{x}, \quad g_{xx} = -\frac{1}{x^2}$$

$$dY_t = \underbrace{\frac{1}{N_t} dN_t}_{\overset{(2)}{=} r\,dt + \alpha\,dB_t} - \underbrace{\frac{1}{2}\frac{1}{N_t^2}(\alpha N_t)^2 dt}_{= \frac{1}{2}\alpha^2 dt}$$

$$\Longrightarrow Y_t = Y_0 + rt + \alpha B_t - \frac{1}{2}\alpha^2 t$$

$$\overset{Y_t = \ln N_t}{\Longrightarrow} N_t = e^{Y_t} = \underline{\underline{\underset{e^{Y_0}}{N_0}\, e^{(r - \frac{\alpha^2}{2})t + \alpha B_t}}}$$

Chk.: If $\mathcal{H}_t = \mathcal{F}_{\{\mathcal{F}_t, \mathcal{F}_{N_0}\}}$, $N_0 \in L^2(\Omega, \mathcal{F}_{N_0}, P)$ indep. of $\mathcal{F}_t$ $\forall t > 0$, then N_t str. sol'n of (2) in $\mathcal{L}^2_{(0,t),\mathcal{H}_t}$ $\forall t > 0$

[Ito formula w. $N_t - N_0 = e^{Y_t} - e^{Y_0} = \int h_x\, dY_t + \frac{1}{2}\int h_{xx} \ldots$ --- $h = e^y$]

$$E(N_t) \overset{(2)}{=} E(N_0) + r\int_0^t E(N_s)\,ds + E\Big(\underbrace{\int_0^t \alpha N_s\,dB_s}_{L^2\text{-Ito-int.},\ E(\cdot) = 0}\Big)$$

$$\frac{d}{dt} \Downarrow \quad Z_t = E(N_t)$$

$$Z_t' = r Z_t, \quad Z(0) = E(N_0)$$

$$\Downarrow$$

$$E(N_t) = Z_t = \underline{E(N_0)\, e^{rt}}$$

Read Ø. Ex. 5.1.1:

$\Downarrow$ law of it. log.

$$N_t \xrightarrow[t\to\infty]{} \infty \quad \text{a.s.} \quad \text{if } r > \tfrac{1}{2}\alpha^2$$

$$N_t \xrightarrow[t\to\infty]{} 0 \quad \text{a.s.} \quad \text{if } r < \tfrac{1}{2}\alpha^2 \quad (\text{but } E(N_t) \to \infty\,!)$$

5.3 Uniqueness

Cauchy problem:

(1) ...

(2) $X_0 = Z$ (r.v.)

Ex. 3: $dX_t = 3X^{\frac{2}{3}}\,dt$, $X_0 = 0$

has ∞ many sol'ns:

$$X_t^a = \begin{cases} 0, & t \le a \\ (t-a)^3, & t \ge a \end{cases} \qquad \forall\, a > 0 \qquad \text{"not Lip"}$$

Need "Lipschitz" to get uniqueness:

(A1) $b: [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$, $\sigma: [0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times m}$ meas.

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le L|x-y|$$

for all $x, y \in \mathbb{R}^n$, $t \in [0,T]$.

Rem. 4:

a) Local Lip. ok in more refined pf's

b) (A1) $\Rightarrow |b(t,x)| + |\sigma(t,x)| \leq \underbrace{|b(t,0)| + |\sigma(t,0)|}_{\text{bad ??}} + 2L|x|$

c) $|\sigma|$ matrix norm, e.g. $|\sigma|^2 = \sum_{i,j} |\sigma_{ij}|^2$

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Also need lin. growth:

(A2) $|b(t,x)| + |\sigma(t,x)| \leq C(1+|x|) \quad \forall t \in [0,T],\ x \in \mathbb{R}^n$

"for existence + $X_t \in L^2$"

and for init. cond'n

(A3) $Z \in L^2(\Omega, P)$, indep. of $\mathcal{F}_\infty^{(n)} := \mathcal{F}_{\{B_s : s \geq 0\}}$

"for $X_t \in L^2$"

Def.: $\mathcal{F}_t^Z := \mathcal{F}_{\{Z, B_s : 0 \leq s \leq t\}}$ "smallest natural filtr. $\supset \mathcal{F}_Z, \mathcal{F}_t$"

Obs. 5: $B_t - B_s$ indep. of $\mathcal{F}_s^Z$ $(s<t) \Rightarrow B_t$ $\mathcal{F}_t^Z$-mart.

Thm. 6: (Sample path uniqueness)

Given $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t^Z\}_t, B_t)$. Assume (A1)–(A3), $X_t, Y_t \in \mathcal{L}^2_{(0,T), \mathcal{F}_t^Z}$ str. sol'n of (1) and (2), then

$$X_t(\omega) = Y_t(\omega) \quad \forall t \in [0,T] \text{ a.s.}$$

Obs. 7:

b) $(A2) + X_t \in \mathcal{L}^2_{(0,T)} \overset{\text{chk}}{\Longrightarrow} \sigma(t, X_t), b(t, X_t) \in \mathcal{L}^2_{(0,T)}$

$\Longrightarrow X_t$ Ito-proc. + msc.

a) A.e. sample paths $t \mapsto X_t(\omega)$ and $t \mapsto Y_t(\omega)$ are equal.

$\rightarrow X_t, Y_t$ indistinguishable (str. than version)

Ex. 8:

$$dX_t = rX_t\,dt + \sigma X_t\,dB_t, \quad B_0 = Z$$

has unique str. sol'n if (A3) holds.

Prop. 9: (A1)-(A3); $X_t, Y_t \in \mathcal{L}^2_{(0,T), \mathcal{F}_t^Z}$ str. sol'ns of (1) and (2).

a) $E|X_t|^2 \leq 3(E|X_0|^2 + TK_1)e^{3K_1 t}, \; t \in [0,T],$

$$K_1 = 2C^2(1+T)$$

b) $E|X_t - Y_t|^2 \leq 3e^{K_2 t} E|X_0 - Y_0|^2, \; t \in [0,T],$

$$K_2 = 3(1+T)L^2$$

Pf. of Thm. 6:

1) $E|X_0 - Y_0|^2 \overset{(2)}{=} 0 \overset{\text{Prop. 9b}}{\Longrightarrow} E|X_t - Y_t|^2 = 0 \;\; \forall t \in [0,T]$

Prop. 9: (A1)-(A3); $X_t, Y_t \in \mathcal{L}^2_{(0,T), \mathcal{F}_t}$ str. sol'ns of (1) ~~and (5)~~. Then

$$E\left(\sup_{t\le T}|X_t - Y_t|^2\right) \le 3e^{K_T T} E|X_0 - Y_0|^2$$

where $K_T = 3(4+T)\cdot L^2$.

Pf. of Thm. 6:

$$E\left(\sup_{t\le T}|X_t - Y_t|^2\right) \overset{\text{Prop. 9}}{\le} C E\underbrace{|X_0 - Y_0|^2}_{\overset{(2)}{=0}} = 0$$

$$\Rightarrow \sup_{t\le T}|X_t - Y_t|^2 = 0 \quad \text{a.s.}$$

$\square$

2) $N_t := \{\omega : X_t \neq Y_t\}$, $\mathbb{Q} \cap [0,T] = \{q_1, q_2, \dots\}$

$$\left(P(\bigcup_{i=1}^{n} N_{q_i}) \le \sum_{i=1}^{n} P(N_{q_i}) = 0\right)$$

$$P\left(\bigcup_{i=1}^{\infty} N_{q_i}\right) = \lim_{n\to\infty} \underbrace{P\left(\bigcup_{i=1}^{n} N_{q_i}\right)}_{= 0 \text{ by } 1)} = 0$$

$$\Rightarrow X_t = Y_t \quad \forall \; t \in \underset{(\text{dense})}{\mathbb{Q} \cap [0,T]} \quad \text{a.s.}$$

$$\overset{X_t, Y_t \text{ cont.}}{\Longrightarrow} \quad X_t = Y_t \quad \forall \; t \in [0,T] \text{ a.s.} \qquad \square$$

$$\div \left[\forall \varepsilon > 0 \; \exists q \in \mathbb{Q} \text{ s.t. } \underbrace{|X_t - X_q|}_{\substack{< \frac{\varepsilon}{2} \text{ a.s.} \\ \text{cont.}}} + \underbrace{|X_q - Y_q|}_{= 0 \text{ a.s.}} + \underbrace{|Y_q - Y_t|}_{\substack{< \frac{\varepsilon}{2} \text{ a.s.} \\ \text{cont.}}} < \varepsilon \text{ a.s.}\right]$$

Must prove Prop. 9!

<u>Lem. 10</u>: Grönwall

$u(t) \ge 0$ cont.; $C_1, C_2 \ge 0$; and

$$u(t) \le C_1 + C_2 \int_0^t u(s)\,ds \quad \forall t \in [0,T]$$

then

$$u(t) \le C_1 e^{C_2 t} \quad \forall \; t \in [0,T]$$

<u>Pf.</u>: Wikipedia (HW) $\square$

<u>Lem 11</u>: $u : [0,T] \to \mathbb{R}^n$, $u \in L^1$, then

$$\left|\int_0^t u \, ds\right|^2 \le t \int_0^t |u|^2 \, ds$$

Pf.: Jensen: $\left(\frac{1}{t}\int_0^t u\,ds\right)^2 \le \frac{1}{t}\int_0^t u^2\,ds$ $\square$

<u>Lem. 12:</u> M_t a.s. t-cont. L^2 submart.

Then $E(\sup|M_t|^2) \leq 4E|M_T|^2$

[Thm. 39 in part 3 (Doob. m. ineq.)]

Pf. of Prop. 9:

1) $$X_t - Y_t \overset{(1)}{=} X_0 - Y_0 + \int_0^t (b(X_s) - b(Y_s))\,ds + \int_0^t (\sigma(X_s) - \sigma(Y_s))\,dB_s$$

$$\Big\Downarrow \ \sup_t,\ E(\cdot) \qquad (a+b+c)^2 \le 3(a^2+b^2+c^2)$$

$$(*)\quad E\Big(\sup_{t\le T} |X_t - Y_t|^2\Big) \le 3E|X_0 - Y_0|^2$$

$$+ 3E\Big(\sup_{t\le T} \Big|\int_0^t (b(X_s) - b(Y_s))\,ds\Big|^2\Big)$$

$$+ 3E\Big(\sup_{t\le T} \Big|\int_0^t (\sigma(X_s) - \sigma(Y_s))\,dB_s\Big|^2\Big)$$

$$= 3E|X_0 + Y_0|^2 + 3I_1 + 3I_2$$

Lem 12: $\overset{\text{a.s. } t\text{-cont.}}{M_t}$ L^2 submart

$E(\sup |M_t|^2) \le 4E|M_T|^2$

(Thm. 39, in part 3)

Doob m-ineq.

$$I_1 \le E\Big(\int_0^T |b(X_s) - b(Y_s)|\,ds\Big)^2$$

$$\overset{\text{Lem. 11}}{\le} T E \int_0^T \underbrace{|b(X_s) - b(Y_s)|^2}_{\le L^2 |X_s - Y_s|^2 \le L^2 \max_{r\le s} |X_r - Y_s|^2} ds$$

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$$I_2 \overset{\text{Lem. 12}}{\le} 4E\Big(\int_0^T (\sigma(X_s) - \sigma(Y_s))\,dB_s\Big)^2$$

$$\overset{\text{Itô-iso.}}{=} 4E\int_0^T \underbrace{|\sigma(X_s) - \sigma(Y_s)|^2}_{\le L^2 |X_s - Y_s|^2 \le L^2 \max_{r\le s} |X_r - Y_r|^2} dt$$

Let $w(t) = E\big(\sup_{r\le t} |X_r - Y_r|^2\big)$, then

$$I_1 + I_2 \le (T+4) L^2 \int_0^T w(t)\,dt$$

and (*) becomes

$$w(T) \leq 3E|X_0 - Y_0|^2 + K_T \int_0^T w(t)\,dt$$

Use Grönwall. $\square$

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Summary:

(1) $X_t = X_0 + \int_0^t b(s,X_s)\,ds + \int_0^t \sigma(s,X_s)\,dB_s$

(2) $X_0 = Z$

Str. sol'n (B_t given)

X_t Ito proc. solving (1) $\forall t \geq 0$ a.s. $\underline{\nabla}$

(A1) $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq L|x-y| \;\forall t \in [0,T]\;\forall x,y$

(A2) $|b(t,x)| + |\sigma(t,x)| \leq C(1+|x|)$ —— " ——

(A3) $Z \in L^2(\Omega, P)$, indep. of $\mathcal{F}_\infty^{(m)}$ $(= \mathcal{F}_{\{B_s : s \geq 0\}})$

$\mathcal{F}_t^Z := \mathcal{F}_{\{Z, B_s : 0 \leq s \leq t\}}$, B_t $\mathcal{F}_t^Z$ - mart.

Uniqueness followed from

Prop. 9: (A1) - (A3), $X_t, Y_t \in \mathcal{L}^2_{(0,T), \mathcal{F}_t^Z}$ str. sol'ns of (1).

$$E\left(\sup_{t\leq T} |X_t - Y_t|^2\right) \leq 3e^{K_T T} E|X_0 - Y_0|^2, \quad t \in [0,T],$$

$$K_T = 3L^2(4+T)$$

5.3 ~~Unique~~ (cont.) ~~of~~ pf. Prop. 9

Lem. 10: $u(t) \leq C_1 + C_2 \int_0^t u(s)\,ds$, $t \in [0,T]$

$$\Downarrow \quad u \geq 0, \text{ cont.}$$

$$u(t) \leq C_1 e^{C_2 t}, \quad t \in [0,T]$$

Lem. 12: M_t cont. L^2 sub-mart.

$$E(\sup_{t \leq T} |M_t|^2) \leq 4 E|M_T|^2$$

Pf of Prop. 9:

1) $$|X_t - Y_t|^2 \overset{(I)}{\leq} 3|X_0 - Y_0|^2 + 3(\int b -)^2 + 3(\int \sigma -)^2$$

$$\Downarrow \quad \sup_t, \text{ then } E(\cdot)$$

$$(*) \quad E(\sup_{t \leq T} |X_t - Y_t|^2) \leq 3E|X_0 - Y_0|^2 + 3I_1 + 3I_2$$

$$I_1 \underset{\text{Jensen}}{\overset{\text{prev.}}{\leq \ldots}} \leq T L^2 \int_0^T E(\max_{r \leq s} |X_r - Y_r|^2) ds$$

$$I_2 = E(\sup_{t \leq T} |\underbrace{\int_0^t (\sigma(X_s) - \sigma(Y_s)) dB_s}_{= M_t, \text{ cont. } L^2\text{-mart.}}|^2)$$

$$\overset{\text{Lem. 12}}{\leq} 4E|\underbrace{\int_0^T (\sigma(X_s) - \sigma(Y_s)) dB_s}_{= M_T}|^2$$

$$\overset{\text{Ito-iso}}{=} 4E\int_0^T |\sigma(X_s) - \sigma(Y_s)|^2 ds$$

$$\overset{(A1)}{\leq} 4L^2 \int \sup_{r \leq s} |X_r - Y_r|^2 ds$$

Let $w(t) = E(\sup_{r \leq t} |X_r - Y_r|^2)$. Then by $(*)$:

3.)

$$w(T) \leq 3E|X_0 - Y_0|^2 + 3L^2(T+4)\int_0^T w(s)\,ds$$

... Grönwall (Lem. 10) ... $\square$

CORRECT
BUT PF. OF
ANOTHER RESULT

~~b) $E|X_t - Y_t|^2$~~

$$\overset{(1)}{=} E\left| X_0 - Y_0 + \int_0^t (b(X_s) - b(Y_s))\,ds + \int_0^t (\sigma(X_s) - \sigma(Y_s))\,dB_s \right|^2$$

Lem. 11, Ito-iso, ...

$$\leq 3E|X_0 - Y_0|^2 + 3E\,t\int_0^t \underbrace{|b(s,X_s) - b(s,Y_s)|^2}_{\overset{(A1)}{\leq} L^2|X_s - Y_s|^2} ds + 3E\int_0^t \underbrace{|\sigma(s,X_s) - \sigma(s,Y_s)|^2}_{\overset{(A1)}{\leq} L^2|X_s - Y_s|^2} ds$$

$u(t) = E|X_t - Y_t|^2$ $\Downarrow$

$$u(t) \leq 3E|X_0 - Y_0|^2 + K_2\int_0^t u(s)\,ds \quad \ldots \text{Grönwall} \ldots \qquad \square$$

5.4 Existence

Ex. 13: $dX_t = X_t^2\,dt, \; X_0 = 1$

$\Rightarrow X_t = \frac{1}{1-t}$ does not ex. after $t = 1$!

Need lin. growth (A2) to have global (in time) ex.!

Thm. 14: (Existence in $\mathcal{L}^2_{(0,T),\mathcal{F}_t^Z}$)

If (A1) - (A3) holds, then there exists a str. sol'n $X_t \in \mathcal{L}^2_{(0,T),\mathcal{F}_t^Z}$ of (1) and (2) and $t \mapsto X_t(\omega)$ is cont. a.s.

Picard - iteration:

$$(3) \begin{cases} Y_t^{(0)} = X_0 = Z \\ Y_t^{(k+1)} = Z + \int_0^t b(s, Y_s^{(k)})\,ds + \int_0^t \sigma(s, Y_s^{(k)})\,dB_s \end{cases}$$

<u>Lem. 15:</u> $E\left(\sup_{t\leq T}|Y_t^{(k+1)}-Y_t^{(k)}|^2\right) \leq C_0 \frac{C_T^k T^k}{k!}$

where $C_T = 2L^2(T+4)$ and $C_0 = 4C^2(T+4)(1+E|Z|)$.

<u>Pf.:</u>

1) $|Y_t^{(k+1)} - Z|^2 \underset{(a+b)^2\leq 2(a^2+b^2)}{\overset{(3)}{\leq}} 2\left|\int_0^t b(Y_s^{(k)})ds\right|^2 + 2\left|\int_0^t \sigma(Y_s^{(k)})dB_s\right|^2$

$\sup_t$, $E(\cdot)$, Jensen, Doob, Ito-iso

$E\left(\sup_{t\leq T}|Y_t^{(k+1)}-Z|^2\right) \leq 2TE\int_0^T |b(Y_s^{(k)})|^2 ds + 2\cdot 4\cdot E\int_0^T |\sigma(Y_s^{(k)})|^2 ds$

$\underset{\text{Fubini}}{\overset{(A2)}{\leq}} 2(T+4)C^2 \underbrace{\int_0^T E(1+|Y_s^{(k)}|)^2 ds}_{\leq T2\left(1+E\left(\sup_{t\leq T}|Y_t^{(k)}|^2\right)\right)} \quad \left(\overset{k=0}{\leq} C_0\right)$

$\Rightarrow E\left(\sup_{t\leq T}|Y_t^{(k)}|^2\right) < \infty \quad \forall\, k\geq 0 \quad (E|Z|^2<\infty)$

2) $|Y_t^{(k+1)} - Y_t^{(k)}|^2 \leq 2\left|\int_0^t (b(Y_s^{(k)}) - b(Y^{(k-1)}))ds\right|^2$

$+ 2\left|\int_0^t (\sigma(Y_t^{(k)}) - \sigma(Y^{(k-1)}))^2 dB_s\right|^2$

--- (A1)

$(**)$ $E\left(\sup_{t\leq T}|Y_t^{(k+1)}-Y_t^{(k)}|^2\right)$

$\leq 2(T+4)L^2\int_0^T E|Y_t^{(k)} - Y_t^{(k-1)}|^2 dt$

Let $w_k(T) = E\left(\sup_{t\leq T}|Y_t^{(k+1)}-Y_t^{(k)}|^2\right)$. By $(**)$,

$w_k(T) \leq C_T\int_0^T w_{k-1}(t)\,dt$

$$\leq C_T^2 \int_0^T \int_0^{t_1} w_{k-2}(t_2)\,dt_2\,dt_1$$

$$\vdots$$

$$\leq C_T^k \int_0^T \int_0^{t_1} \cdots \int_0^{t_{k-1}} \underbrace{w_0(t_k)}_{1)}\,dt_k \cdots dt_1$$

Since $w_0(t) = E(\sup_{t\leq T} |Y_t^{(1)} - \underset{\overset{\shortparallel}{Z}}{Y_t^{(0)}}|^2) \leq C_0$,

$$w_k(T) \leq C_0 C_T^k \int_0^T \cdots \int_0^{t_{k-1}} dt_k \cdots dt_1 = C_0 \frac{C_T^k T^k}{k!} \quad \square$$

<u>Pf. of Thm. 14:</u>

1) <u>Existence</u>

$$V = L^2(\Omega; C_b([0,T])) \text{ w. } \|\varphi\|_V^2 = \underset{\overset{\shortparallel}{E(\sup_{t\leq T}|\varphi|^2)}}{\int_\Omega (\sup_{t\leq T}|\varphi(t,\omega)|)^2 dP(\omega)}$$

is a complete metric sp. (Banach)

$$\|Y_t^{(m)} - Y_t^{(n)}\|_V = \|\sum_{k=m}^{n-1} (Y_t^{(k+1)} - Y_t^{(k)})\|_V$$

$$\leq \sum_{k=m}^{\infty} \|Y_t^{(k+1)} - Y_t^{(k)}\|_V \overset{\text{Lem. 15}}{\leq} \sum_{k=m}^{\infty} \left(C_0 \frac{C_T^k T^k}{k!}\right)^{\frac{1}{2}}$$

$$\xrightarrow[m\to\infty]{} 0 \quad \text{(ratio test)}$$

Hence $\{Y_t^{(k)}\}$ Cauchy in V and exists V-limit X_t (V complete).

2) <u>$X_t \in \mathcal{L}^2_{(0,T),\mathcal{F}_t^Z}$, t-cont. version a.s.</u>

$$Y_t^{(k)} \to X_t \quad \text{in } V$$

$$\Rightarrow \sup_{t\leq T} |Y_t^{(k)} - X_t| \to 0 \quad \text{in } L^2(\Omega)$$

$$\Rightarrow \sup_{t \in T} |Y_t^{(k_l)} - X_t| \to 0 \quad \text{a.s.}$$

subseq'nce $\leftarrow$ (X_t a.s. t-cont. since a.s. unif. limit of a.s. t-cont. $Y_t^{(k_l)}$ (Ito-proc)

Since $Y_t^{(k_l)} \to X_t \ \forall t \geq 0$ a.s., X_t $\mathcal{F}_t^Z$-adap.

and jointly meas. since this holds for $Y_t^{(k_l)}$.

$$E|X_t|^2 \leq E(\sup_{t \leq T} |X_t|^2) = \|X_t\|_V^2 < \infty, \quad t \in [0,T]$$

$$\Rightarrow X_t \in \mathcal{L}^2_{(0,T), \mathcal{F}_t^Z}$$

3) <u>X_t satisfy (1) a.s. $\forall t \geq 0$</u>

$$(***) \quad E\int_0^T |Y_t^{(k)} - X_t|^2 \, dt \leq T\, E(\sup_{t \leq T} |Y_t^{(k)} - X_t|^2) \xrightarrow[k \to \infty]{V\text{-conv.}} 0$$

Then for $\forall t \in [0,T]$:

$$E\left(\int_0^t (b(Y_t^{(k)}) - b(X_t))\, ds\right)^2 \overset{\text{Jensen}}{\leq} t \cdot E\int_0^t |b - b|^2 \, ds \xrightarrow[(***)]{(A1)} 0$$

$$E\left(\int_0^t (\sigma(Y_t^{(k)}) - \sigma(X_t))\, dB_s\right)^2 = E\int_0^t |\sigma - \sigma|^2 \, ds \xrightarrow[(***)]{(A1)} 0$$

Hence $\forall t \geq 0 \ \exists$ subseq'nce $Y_t^{(k_l)}$ s.t.

(i) int's conv. a.s. $\forall t \in [0,T]$ (from L^2 conv)

(ii) $Y_t^{(k_l)} \to X_t$ a.s. $\forall t \in [0,T]$

Set $k = k_l$ in (3) and pass to the limit:

$$X_t = Z + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, dB_s \quad \text{a.s.} \ \square$$