EXERCISES

09.01. - 14.01.

Exercise 1. Let $\Lambda = k[X]/(X^2 - 1)$.
Find a quiver Q with admissible relations R such that $\Lambda \cong kQ/(R)$.
HINT: The result depends on the characteristic of k.

Exercise 2. Let $e \in \Lambda$ be a central idempotent.
Show that $\Lambda \cong e\Lambda e \times (1-e)\Lambda(1-e)$ as algebras.

16.01.-21.01.

Exercise 3. Let Λ be a finite dimensional algebra.
Convince yourself that $\mathrm{D}\Lambda$ is a Λ-bimodule.
Show that $\nu = - \otimes_\Lambda \mathrm{D}\Lambda$ and $\nu^- = \mathrm{Hom}_\Lambda(\mathrm{D}\Lambda, -)$.

Exercise 4. Let Λ be given by the quiver

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[loop right, "\beta"] \end{tikzcd}$$

and relations $\beta\alpha$ and β^2.

- Find all indecomposable projective and all indecomposable injective modules.
- For any indecomposable injective module I, calculate τI, $\tau^2 I$, $\dots$.

23.01.-28.01.

Exercise 5. For Λ as in Exercise 4, calculate all almost split sequences starting or ending in the indecomposable modules calculated there.

Exercise 6. Let Λ be a finite dimensional algebra.

- Let P indecomposable projective, and $X \rightarrowtail E \twoheadrightarrow \tau^- X$ an almost split sequence. Show that P is isomorphic to a direct summand of E if and only if X is isomorphic to a direct summand of $\mathrm{Rad}P$.
- *(maybe a bit harder?)* Let P be indecomposable projective and injective, but not simple. Show that there is an almost split sequence
$$\mathrm{Rad}P \rightarrowtail P \oplus \frac{\mathrm{Rad}P}{\mathrm{Soc}P} \twoheadrightarrow \frac{P}{\mathrm{Soc}P}.$$
In particular $\mathrm{Rad}P = \tau(P/\mathrm{Soc}P)$.

30.01.-04.02.

Exercise 7. Let Λ be given by the quiver

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \end{tikzcd}$$

and relation $\alpha\beta$.
Find the Auslander-Reiten quiver of Λ.

Exercise 8. Let M be an indecomposable Λ module.

Show that $\mathrm{Rad}^2(M,M) = \mathrm{Rad}^2\mathrm{End}(M)$ if and only if $\mathrm{End}(M)$ is a skewfield.

06.02.-11.02.

Exercise 9. Let Λ be given by the quiver

$$\begin{tikzcd} & 2 \arrow[dl, "\beta"'] & \\ 3 \arrow[d, "\gamma"'] & & 1 \arrow[ul, "\alpha"'] \\ 4 \arrow[r, "\delta"'] & 5 \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

and relations $\beta\alpha$ and $\delta\gamma\beta$.

Find the Auslander-Reiten quiver of Λ.

Exercise 10. Let Λ be given by the quiver

$$\begin{tikzcd} & 2 \arrow[dl, "\alpha"'] \arrow[dr, "\gamma"] & \\ 3 \arrow[dr, "\beta"'] & & 1 \arrow[dl, "\delta"] \\ & 4 \arrow[uu, "\epsilon"] & \end{tikzcd}$$

and relations $\epsilon\beta$, $\epsilon\delta$, $\beta\alpha - \delta\gamma$, and $\beta\alpha\epsilon$.

Find the Auslander-Reiten quiver of Λ.

13.02.-18.02.

Exercise 11. Let Λ be a finite dimensional representation finite algebra, and Γ be its Auslander algebra.

(1) Show that Γ^{op} is the Auslander algebra of Λ^{op}.
(2) Show that the following are equivalent:
  - Λ is semi-simple;
  - Γ is semi-simple;
  - $\mathrm{gl.dim}\Gamma \leqslant 1$.

Exercise 12. Let $\Lambda = k[1 \longrightarrow 2 \longleftarrow 3]$.

- Compute the AR-quiver of Λ. In particular show that Λ is representation finite.
- Let Γ be the Auslander algebra of Λ. Find a presentation of Γ as quiver with relations.
- Compute the AR-quiver of Γ.

20.02.-25.02.

Exercise 13. Calculate the Auslander-Reiten quiver of

(1) kQ, with $Q = \begin{tikzcd} & 4 \arrow[d] & \\ 1 \arrow[r] & 2 & 3 \arrow[l] \end{tikzcd}$

(2) $kQ/(R)$, with $Q = \begin{tikzcd} & 2 \arrow[dl, "\beta"'] & \\ 3 \arrow[d, "\gamma"'] & & 1 \arrow[ul, "\alpha"'] \\ 4 \arrow[r, "\delta"'] & 5 \arrow[ur, "\epsilon"'] & \end{tikzcd}$ and $R = \{\alpha\epsilon\delta, \beta\alpha\epsilon, \gamma\beta\alpha, \delta\gamma\beta, \epsilon\delta\gamma\}$.

Exercise 14. For a quiver Q, consider the following procedure, which we will call mutation: pick a sink or source i, and reverse all arrows ending or starting in i (i.e. making i a source or sink, respectively.)

We call two quivers equivalent if one can be obtained from the other by repeatedly doing the above. Observe that mutation does not change the underlying graph of Q.

For all Euclidean diagrams, determine the equivalence classes of quivers with underlying graph equal to this diagram.

27.02.-04.03.

Exercise 15. (1) For $Q = [1 \longrightarrow 2]$, calculate the matrix Φ of the Coxeter transformation. Calculate Φ^3.
(2) For $Q = [\ 1 \rightrightarrows 2\ ]$, calculate the matrix Φ of the Coxeter transformation. Calculate Φ^n for all positive n. Calculate the dimension vectors of $\tau^n I_1$ and $\tau^n I_2$ for all n.

Exercise 16. For $Q = [\ 1 \rightrightarrows 2\ ]$, determine the isomorphism classes of kQ-modules of dimension vector $(1,1)^t$.

06.03.-11.03.

Exercise 17. Let $\Lambda = kQ/(R)$, with

$$Q = \begin{tikzcd} 1 \arrow[r, "a"] \arrow[d, "c"'] & 2 \arrow[r, "b"] \arrow[d, "d"'] & 3 \arrow[d, "e"'] \\ 4 \arrow[r, "f"] & 5 \arrow[r, "g"] & 6 \end{tikzcd} \quad \text{and } R = \{da - fc, eb - gd\}.$$

Calculate the Auslander-Reiten quiver of Λ.

Exercise 18. Consider the algebras $\Lambda = k[\ 1 \overset{x}{\underset{y}{\rightrightarrows}} 2\ ]$ and $\Gamma = k[x,y]/(x^2, y^2)$.

Motivation: The idea of this exercise is that we know (or soon will know) the representation theory of Λ quite well, and we want to transfer this knowledge to Γ.

Convince yourself that there is a morphism of algebras

$$\Gamma \longrightarrow \Lambda : 1 \longmapsto 1; x \longmapsto x; y \longmapsto y,$$

and thus an associated restriction functor $\texttt{res}\colon \operatorname{mod}\Lambda \longrightarrow \operatorname{mod}\Gamma$.

Show that

- If M is indecomposable in $\operatorname{mod}\Lambda$, then $\texttt{res}M$ is indecomposable in $\operatorname{mod}\Gamma$.
- If M and N are indecomposable, and not isomorphic, then neither are $\texttt{res}M$ and $\texttt{res}N$ – unless $\{M,N\} = \{S_1, S_2\}$.
- If X is indecomposable in $\operatorname{mod}\Gamma$, then there is an indecomposable M in $\operatorname{mod}\Lambda$, such that $X = \texttt{res}M$ – unless $X = \Gamma$.

13.03.-18.03.

Exercise 19. Let Q as in Exercise 16. For any two indecomposable modules M and N of dimension vector $(1,1)^t$, calculate $\operatorname{Ext}^1(M,N)$.

Exercise 20. Let $\Lambda = k[\ 1 \overset{\textstyle\rightarrow}{\rightrightarrows} 2\ ]$. Let Γ be any finite dimensional k-algebra. Show that there is a fully faithful functor $\operatorname{mod}\Gamma \longrightarrow \operatorname{mod}\Lambda$.

Remark: This means that the algebra Λ is *strictly wild*.

Strategy:

- Denote by findim R the category of finite dimensional modules over a not necessarily finite dimensional algebra R.
- Denote by $k\langle x_1, \dots, x_n\rangle$ the free algebra in n non-commuting variables.
- Find a surjective algebrahomomorphism $k\langle x_1, \dots, x_n\rangle \twoheadrightarrow \Gamma$ for some n – this induces a fully faithful restriction functor.
- Consider the algebra homomorphism

$$k\langle x, y\rangle \longrightarrow \mathrm{Mat}_{n+2}(k\langle x_1, \dots, x_n\rangle)$$
$$x \longmapsto \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ x_1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & x_2 & 1 & & 0 & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & x_n & 1 & 0 \end{pmatrix}$$
$$y \longmapsto \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

  Show that the composition

$$\mathrm{findim}\, k\langle x_1, \dots, x_n\rangle \underset{\text{Morita}}{\cong} \mathrm{findim}\, \mathrm{Mat}_{n+2}(k\langle x_1, \dots, x_n\rangle) \xrightarrow{\text{res}} \mathrm{findim}\, k\langle x, y\rangle$$

  is fully faithful.
- Find a fully faithful functor findim $k\langle x, y\rangle \longrightarrow \mathrm{mod}\,\Lambda$.

20.03.-25.03.

Exercise 21. Easy in principle, but a lot of linear algebra – you might want to use Maple or something.

Consider the two quivers Q_2 and Q_3 from Exercises 16 and 20, respectively, and the associated Coxeter transformations Φ_2 and Φ_3.

For which $\mathbf{v} \in \mathbb{Z}^2_{\geqslant 0}$ does one have $\Phi_2^n \mathbf{v} < \mathbf{0}$ for $n \gg 0$? What about for $n \ll 0$? What about the same two questions for Φ_3?

Draw pictures. Explain what this has to do with preprojective, preinjective and regular representations.

Exercise 22. For Q_3 as above, find a short exact sequence containing precisely two regular terms.

Hints: Use the previous exercise. You may for instance find such a sequence where the three mudules have dimension vectors $(0,1)^t$, $(1,2)^t$, and $(1,1)^t$, respectively.

27.03.-01.04.

Exercise 23. Let $\Lambda = k[1 \longrightarrow 2 \longrightarrow 3]$, and let T be the direct sum of the (right) modules $S_1 = P_1$, $P_3 = I_1$, and $S_3 = I_3$ (as in the lecture).

Let $\Gamma = \mathrm{End}_\Lambda(T)$.

(1) Find a presentation of Γ as quiver with relations.
(2) Draw the AR-quivers of Λ and Γ.
(3) Find the subcategories $\mathscr{T}$ of $\mathrm{mod}\,\Lambda$ and $\mathscr{Y}$ of $\mathrm{mod}\,\Gamma$.
(4) Find the subcategories $\mathscr{F}$ of $\mathrm{mod}\,\Lambda$ and $\mathscr{X}$ of $\mathrm{mod}\,\Gamma$.

Exercise 24. Let $T \in \mathrm{mod}\,\Lambda$ such that

(1) $\mathrm{proj.\,dim}\, T \leqslant 1$;

(2) $\mathrm{Ext}^1_\Lambda(T, T) = 0$;

(3) for some n, there is an exact sequence

$$0 \longrightarrow \Lambda \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^n \longrightarrow 0$$

with $T^i \in \operatorname{add} T$.

Show that T is a classical tilting module.

03.04.-08.04.

Exercise 25. Let Λ be a basic finite dimensional algebra, S a simple Λ-module such that

- $\operatorname{Ext}^1_\Lambda(S, S) = 0$ (i.e. there is no loop at the corresponding vertex in the quiver of Λ), and
- $\operatorname{Hom}_\Lambda(D\Lambda, S) = 0$.

Let $T = \Lambda/P(S) \oplus \tau^- S$, where $P(S)$ denotes the indecomposable projective corresponding to S.

Show that T is tilting.

Exercise 26. Let $\Lambda = k\left[\begin{tikzcd}[row sep=small, column sep=small] & & 6 & & \\ 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[u] \arrow[r] & 4 \arrow[r] & 5 \end{tikzcd} \right]$, and let T be the direct sum of the (right) modules P_1, P_5, P_6, $\tau^{-3}P_1$, $\tau^{-3}P_5$, and $\tau^{-2}P_6$.

Let $\Gamma = \operatorname{End}_\Lambda(T)$.

(1) Draw the AR-quivers of Λ. Hints: Dimension vectors should suffice, we don't need the explicit representations. There are 36 indecomposables.
(2) Prove that T is a tilting module.
(3) Find a presentation of Γ as quiver with relations.
(4) Find the AR-quiver of Γ. Hint: Exercise 17.
(5) Find the subcategories $\mathscr{T}$ and $\mathscr{F}$ of $\operatorname{mod}\Lambda$ and $\mathscr{X}$ and $\mathscr{Y}$ of $\operatorname{mod}\Gamma$.