Norwegian University of Science and Technology
Department of Mathematical Sciences

MA8404 Numerical solution of time dependent differential equations
Autumn 2017

Exercise set 5

You may very well use some kind of software like Maple or Mathematica to solve these problems. But if you do, put some efforts into making the answers as simple and readable as possible.

1 In this exercise, consider methods given by:

$$\begin{array}{c|cc} \gamma & \gamma & 0 \\ c_1 & a_{21} & \gamma \\ \hline 0 & b_1 & b_2 \end{array}$$

a)
- Find all 2nd order methods. (use c_2 and γ as free parameters).
- Find all 3th order methods.
- Find all 2nd order stiffly accurate methods.

b) Show that the stability function $R(z)$ for this method is given by

$$R(z) = \frac{P(z)}{(1-\gamma z)^2}.$$

What is $P(z)$ for the methods (all order 2, order 3, stiffly accurate) from point a).

c) For which γ's is the method A-stable? Are the 3th order and/or the stiffly accurate methods A-stable?

d) Plot the stability regions for the method for some different values of γ. Use some appropriate software.

e) *(Optional)*
Since the linear test problem $y' = \lambda y$ has $y(t+h) = e^{h\lambda}y(t)$ as exact solution, the stability function $R(z)$ is a rational approximation to the function e^z. Will the numerical solution grow faster or slower than the exact solution, that is: when will $|R(z)| > |e^z|$ (or vice versa).
To answer this questions, plot

$$A = \{z \in \mathbb{C}; \qquad |R(z)| > |e^z|\}$$

The domain A is called an *order star*, you may understand why when observing the plots. (see SODEII IV.4 for more).

2 **a)** Prove that a method with an explicit first stage and $b_1 \neq 0$ can not be algebraically stable.

b) Are any of the methods discussed in Problem 1 B-stable?

3 Find the index and the exact solution of the following DAE:

$$\begin{aligned} y_1' &= y_1 & y_1(0) &= 1 \\ y_2' &= z - y_2, & y_2(0) &= 0 \\ z' &= z + y_2 - 2w & z(0) &= 1 \\ 0 &= y_1 - \exp(y_2), & w(0) &= 0 \end{aligned}$$

4 Given the linear DAE

$$\begin{pmatrix} 0 & 0 \\ 1 & \eta t \end{pmatrix} \begin{pmatrix} y' \\ z' \end{pmatrix} + \begin{pmatrix} 1 & \eta t \\ 0 & 1+\eta \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$$

where f, g are two given, smooth functions, and $\eta \in \mathbb{R}$ is a constant.

a) Find the exact solution of the problem.

b) Apply the implicit Euler method to the problem, and show that numerical solution will not converge to the exact solution for all values of η.

c) Transform the problem by introducing a new set of variables: $\hat{y} = y + \eta t z$, $\hat{z} = z$, and apply the implicit Euler on the transformed problem. Will the numerical solution converge to the exact solution now?

5 Consider the three equations

$x' = y - ax^2 + \cos t,$	$y = ax^2$	Index 1 DAE
$x' = \cos t$		State space form
$x' = y - ax^2 + \cos t,$	$y' = 2ax(y - ax^2 + \cos t)$	ODE

of which the first is our originally index 1 DAE, the second is the state space form, and the last is the ODE obtained from differentiating the algebraic constraint of the original problem, sometimes called the underlying ODE. The aim of this exercise is to demonstrate that solving the underlying ODE rather than the original problem is not necessarily a good idea.

Use some DAE solver (I have used MATLABs ODE15s) and solve the three problems over the interval $[0, 10\pi]$ with $x(0) = y(0) = 0$.

a) Solve the three problems using e.g. $a = 1$, $a = 10$ and $a = 100$, using the same tolerances (e.g. the default tolerances in MATLAB.) Comment on the result.

b) For $a = 100$, adjust the tolerances such that the error at the end of the interval is approximately 10^{-6}. How much compational work in terms of number of steps, number of function evaluations and number of jabobians do you need in each case?

c) Discuss the stability of the problems in terms of eigenvalues of the Jacobians on the constraint given by $y = ax^2$.