Norwegian University of Science and Technology
Department of Mathematical Sciences

MA8404 Numerical solution of time dependent differential equations
Autumn 2017

Exercise set 6

$\boxed{1}$ Given the stochastic differential equation:

$$\mathrm{d}X = -\alpha X\mathrm{d}t + \sigma \mathrm{d}W(t), \qquad X(0) = x_0 \tag{1}$$

where $\alpha, \sigma \in \mathbb{R}$ are some constants, $\alpha > 0$. For simplicity, we also assume that the initial value x_0 is a constant (and not a stochastic variable).

a) Prove that the exact solution of this SDE is

$$X(t) = \mathrm{e}^{-\alpha t}x_0 + \beta \int_0^t \mathrm{e}^{-\alpha(t-s)}\mathrm{d}W(s)$$

Also, show that this is a Gaussian process, with

$$\mathbb{E}X(t) = \mathrm{e}^{-\alpha t}x_0, \qquad \mathrm{Var}X(t) = \frac{\sigma^2}{2\alpha}(1 - \mathrm{e}^{-2\alpha t}).$$

b) When (1) is solved numerically by an Euler-Maruyama method, the strong mean square order of the numerical solution is 1, and not 1/2 as expected. Explain why.

Given a Wiener process simulated by

$$W(t_n + h) = W(t_n) + \xi_n\sqrt{h}, \qquad \xi_n \sim \mathcal{N}(0, 1).$$

Then the exact solution of (1) is given by

$$X(t_n + h) = \mu X(t_n) + \xi_n \kappa.$$

where

$$\mu = \mathrm{e}^{-\alpha h}, \qquad \kappa^2 = \frac{\sigma^2}{2\alpha}(1 - \mathrm{e}^{-2\alpha h}).$$

c) Solve (1) by the Euler-Maruyama method, and compare the exact and the numerical solution. Verify the order result from point **b)** numerically.

Use e.g. $\alpha = 1$ and $\sigma = 0.5$ in your experiments.

$\boxed{2}$ Given an Itô SDE

$$\mathrm{d}X = f(X)\mathrm{d}t + g(X)\mathrm{d}W(t), \qquad X(0) = x_0$$

and the SRK method

$$\begin{aligned} H_2 &= Y_n + \sqrt{h} g(Y_n) \\ Y_{n+1} &= Y_n + hf(Y_n) + I_1 g(Y_n) + \frac{I_{11}}{\sqrt{h}}(g(H_2) - g(Y_n)) \end{aligned}$$

where

$$I_1 = \int_{t_n}^{t_{n+h}(\mathrm{d}W(s)} = \Delta W_n, \qquad I_{11} = \int_{t_n}^{t_n+h} \int_{t_n}^{t_n+s} \mathrm{d}W(s_1)\mathrm{d}W(s) = \frac{1}{2}(\Delta W_n^2 - h).$$

a) Write up the stochastic Butcher tablaux for the method, and find its strong order.

b) Discuss the MS-stability properties of the method, applied to the linear test equation

$$dX = \lambda X \mathrm{d}t + \mu X \mathrm{d}W(t)$$

with λ, μ being real constants.