How to read order conditions from a rooted tree.

Given an s-stage RK-method:

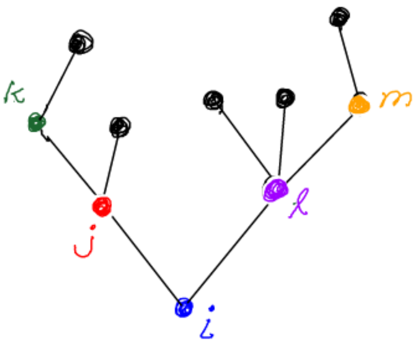
$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ \vdots & & \vdots & \\ & a_{s1} & \cdots & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array} \quad , \quad \text{with} \quad c_i = \sum_{j=1}^{s} a_{ij}$$

For a given tree τ, there is a corresponding order condition

$$\hat{\phi}(\tau) = \frac{1}{\gamma(\tau)}$$

and the procedure to find $\hat{\phi}$ and γ is as follows.

Take some rooted tree (the $\bullet$ is the root)



τ

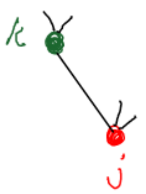
The order $\rho(\tau)$ is the number of nodes, so in this case

$$\rho(\tau) = 10$$

Label all but the terminal nodes, with indices $i, j, k, \ldots$ starting with the root, and move upwards along each branch.

Form a product of the coefficients:

b_i for the root



a_{jk} for each edge between the nodes j and k, j being closest to the root.

c_ℓ for each terminal node.

$\hat{\phi}$ is then the sum of these products, over all indices from 1 to s. So, in the example

$$\hat{\phi}(\tau) = \sum_{i=1}^{s} \sum_{j=1}^{s} \sum_{k=1}^{s} \sum_{\ell=1}^{s} \sum_{m=1}^{s} b_i \, a_{ij} \, c_j \, a_{jk} \, c_k \, a_{i\ell} \, c_\ell^2 \, a_{\ell m} \, c_m$$

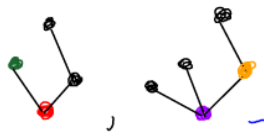
The integer $\gamma(\tau)$ is found recursively:

Let $\tau_1, \dots, \tau_\varkappa$ be the subtrees you have if the root of τ is removed, thus

or $\tau = [\tau_1, \tau_2, \dots, \tau_\varkappa]$

Then $\gamma(\tau) = \rho(\tau) \cdot \prod_{k=1}^{\varkappa} \gamma(\tau_k)$, with $\gamma(\bullet) = 1$

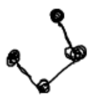
For the example above we get:

$\tau = [$  $]$

and we first have to find γ for each of those (colors ignored)

 $= [\bullet] \Rightarrow \gamma(\ ) = 2 \cdot 1 = 2$

$\rho(\ ) = 2$

 $= [\bullet, \ ] \Rightarrow \gamma(\ ) = 4 \cdot 1 \cdot 2 = 8$

$\rho(\ ) = 4$

 $= [\bullet, \bullet, \ ] \Rightarrow \gamma(\ ) = 5 \cdot 1 \cdot 1 \cdot 2 = 10$

$\rho(\ ) = 5$

So $\gamma(\tau) = \rho(\tau) \cdot \gamma(\ ) \cdot \gamma(\ ) = 10 \cdot 8 \cdot 10 = 800.$