(1) Beregn verdien av integralet

$$\int_0^1 \frac{\sin x}{x}\,dx$$

med en nøyaktighet bedre enn 10^{-4}.

$\frac{\sin x}{x}$ har ingen elementær anti-derivert.

Vi erstatter derfor $\sin x$ med et Taylor-polynom : (utvikler om 0)

$$f(x) = \sin x \qquad f(0) = 0$$

$$f'(x) = \cos x \qquad f'(0) = 1$$

$$f''(x) = -\sin x \qquad f''(0) = 0$$

$$f'''(x) = -\cos x \qquad f'''(0) = -1$$

$$P_n(x) = f(a) + f'(a)\cdot x + \frac{f''(a)}{2!}(x)^2 + \dots + \frac{f^{(n)}(a)}{n!}x^n$$

$$= 0 + 1\cdot x + \frac{0}{2!}x^2 - \frac{x^3}{3!} + \dots + (-1)^n$$

$$\frac{P_{2n}(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots + \frac{(-1)^n\, x^{2n}}{(2n+1)!}$$

$$\int_0^1 \frac{\sin x}{x}\,dx = \int_0^1 \frac{P_{2n}(x)}{x}\,dx + \int_0^1 \frac{R_{2n}(x)}{x}\,dx$$

$$R_{2n}(x) = \frac{f^{(2n+1)}(z)}{(2n+1)!} x^{2n+1}$$

$$|R_{2n}(x)| = \frac{|f^{(2n+1)}(z)|}{(2n+1)!} x^{2n+1} \leq \frac{1}{(2n+1)!} x^{2n+1}$$

$$\left| \int_0^1 \frac{R_{2n}(x)}{x} dx \right| \leq \int_0^1 \left| \frac{R_{2n}(x)}{x} \right| dx \leq \int_0^1 \frac{1}{(2n+1)!} \frac{x^{2n+1}}{x} dx$$

$$= \frac{1}{(2n+1)!} \int_0^1 x^{2n} dx$$

$$= \frac{1}{(2n+1)!} \left[\frac{1}{2n+1} x^{2n+1} \right]_0^1$$

$$= \frac{1}{(2n+1)!(2n+1)} .$$

Vil ha restleddet mindre enn 10^{-4} :

$$\left| \int_0^1 \frac{R_{2n}(x)}{x} dx \right| \leq \frac{1}{(2n+1)!(2n+1)} < 10^{-4}$$

$$10^4 < (2n+1)!(2n+1)$$

Oppfylt for $n > 2$.

For $n = 3$, får vi $P_{2 \cdot 3}(x) = P_6(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$$\int_0^1 \frac{\sin x}{x}\,dx = \int_0^1 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!}\right)dx$$

$$= \left[x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!}\right]_0^1 = 1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!}$$

$$= \frac{1703}{1800} \approx \underline{\underline{0,9461}}$$

[redacted]

(2) Finn summen av

$$\sum_{n=1}^{\infty} \frac{(-\frac{1}{2})^{n+1}}{n(n+1)}$$

La $\displaystyle\sum_{n=1}^{\infty} \frac{(-x)^{n+1}}{n(n+1)} = S(x)$ for $x \in (-R, R)$.

Deriverer m.h.p. x :

$$\sum_{n=1}^{\infty} \frac{-(-x)^n}{n} = \frac{d}{dx} S(x)$$

$$\sum_{n=1}^{\infty} \frac{(-1)(-1)^n x^n}{n} = \frac{d}{dx} S(x)$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = \frac{d}{dx} S(x)$$

$$\ln(1+x) = \frac{d}{dx} S(x)$$

$$\int \ln(1+x)\,dx = S(x)$$

$$(1+x)\ln(1+x) - (1+x) + C = S(x)$$

$$(1+x)\ln(1+x) - x + C_1 = S(x)$$

$$S(0) = 0 \Rightarrow C_1 = 0$$

$$S(x) = (1+x)\ln(1+x) - x$$

Vår rekke er $S(\frac{1}{2})$.

Konvergensintervall ?

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \frac{(-x)^{n+2}}{(-x)^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)(n+2)}}{\frac{1}{n(n+1)}} (-x) \right|$$

$$= \lim_{n\to\infty} \left| \frac{n(n+1)}{(n+1)(n+2)} \right| \cdot |x|$$

$$= |x|$$

Forholdstesten :

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = a \quad : \quad \begin{array}{ll} a<1 & \text{konvergent} \\ a>1 & \text{divergent} \\ a=1 & \text{Ingen konklusjon} \end{array}$$

Hvis vi krever $|x|<1 \Rightarrow$ konvergens.

Det vil si : $x \in (-1,1)$ gir oss en konvergent rekke.

Siden $\frac{1}{2} \in (-1,1)$ vil :

$$S\left(\tfrac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)^{n+1}}{n(n+1)} = \left(1+\tfrac{1}{2}\right)\ln\left(1+\tfrac{1}{2}\right) - \tfrac{1}{2}$$

$$= \frac{3}{2}\ln\frac{3}{2} - \frac{1}{2} .$$