5 a) $\sum_{n=1}^{\infty} \frac{x^{n+2}}{n(n+2)4^n}$

Konvergensradius :

$$\lim_{n\to\infty} \left| \frac{\frac{x^{(n+1)+2}}{(n+1)((n+1)+2)4^{n+1}}}{\frac{x^{n+2}}{n(n+2)4^n}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+3}}{x^{n+2}} \cdot \frac{n(n+2)4^n}{(n+1)(n+3)4^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| |x| \frac{n(n+2)}{(n+1)(n+3)\cdot 4} \right|$$

$$= \frac{|x|}{4} \lim_{n\to\infty} \frac{n(n+2)}{(n+1)(n+3)}$$

$$= \frac{|x|}{4}$$

For de verdier av x som gjør at dette forholdet blir < 1, får vi konvergens (jfr. forholdstesten).

$$\frac{|x|}{4} < 1 \iff |x| < 4 \iff x \in (-4, 4).$$

Konvergensintervall : $(-4,4)$

Konvergensradius : 4 (halve lengden av konvergensintervallet).

$x = -4$: $\sum_{n=1}^{\infty} \frac{(-4)^{n+2}}{n(n+1)4^n} = \sum_{n=1}^{\infty} \frac{(-4)^n 4^2}{n(n+1)4^n} = 16 \sum \frac{1}{n(n+1)} \cdot \frac{(-4)^n}{4^n}$

$$= 16 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left(\frac{-4}{4}\right)^n = 16 \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}$$

Rekken er alternerende.

$$|a_n| = \frac{1}{n(n+1)} > \frac{1}{(n+1)(n+2)} = |a_{n+1}| \qquad \text{(avtagende)}$$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n(n+1)} = 0$$

Rekken konvergerer iflg. alternerende rekkes test.

$$x = 4: \quad \sum_{n=1}^{\infty}{}' \frac{4^{n+2}}{n(n+1)\,4^n} = 16 \sum_{n=1}^{\infty}{}' \frac{1}{n(n+1)}$$

Grensesammenligningstesten:

$$\lim_{n\to\infty} \frac{\frac{1}{n(n+1)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n(n+1)} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

Rekken konvergerer siden $1 < \infty$.

Altså: vi har konvergens for $x \in [-4, 4]$.

b) Vi har $g(x) = \sum_{n=1}^{\infty} \frac{x^{n+2}}{n(n+2)4^n}$

$|x| < R$:

$$g'(x) = \sum_{n=1}^{\infty} \frac{x^{n+1}}{n4^n}$$

$$= x \sum_{n=1}^{\infty} \frac{x^n}{n4^n}$$

$$= x \sum_{n=1}^{\infty} \frac{\left(\frac{x}{4}\right)^n}{n}$$

$$= x \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n \left(\frac{x}{4}\right)^n}{n}$$

$$= x \sum_{n=1}^{\infty} \frac{(-1)^n \left(-\frac{x}{4}\right)^n}{n}$$

$$= x \sum_{n=1}^{\infty} \frac{-(-1)^{n-1} \left(-\frac{x}{4}\right)^n}{n}$$

$$= -x \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \left(-\frac{x}{4}\right)^n}{n}$$

$$= -x \ln\left(1 + \left(-\frac{x}{4}\right)\right)$$

$$= -x \ln\left(1 - \frac{x}{4}\right).$$

(6) $$\int_0^1 \sqrt{1+t^4}\,dt = \int_0^1 \left(1+\frac{t^4}{2} - \frac{1}{8}\,\frac{t^8}{(1+z)^{3/2}}\right) dt$$

$$= \left[t+\frac{t^5}{10}\right]_0^1 - \int_0^1 \frac{t^8}{8(1+z)^{3/2}}\,dt$$

$$= \left(1+\frac{1}{10}\right) - \frac{1}{8(1+z)^{3/2}} \int_0^1 t^8\,dt$$

$$= 1{,}1 - \frac{1}{8(1+z)^{3/2}} \left[\frac{t^9}{9}\right]_0^1$$

$$= 1{,}1 - \frac{1}{72(1+z)^{3/2}} \,.$$

Siden $z \in [0,x]$ og $x \in [0,1]$, har vi at z tar verdier mellom 0 og 1.

Vi skal finne største og minste verdi for $\frac{1}{72(1+z)^{3/2}}$ (en avtagende funksjon!).

$z=0$: $\frac{1}{72(1+0)^{3/2}} = \frac{1}{72}$.

$z=1$: $\frac{1}{72\cdot(1+1)^{3/2}} = \frac{1}{72\cdot 2^{3/2}} = \frac{1}{72\cdot 2\cdot\sqrt{2}} = \frac{1}{144\sqrt{2}}$.

Så hvis $R = \frac{1}{72(1+z)^{3/2}}$ og $z\in[0,1]$, må

$\frac{1}{144\sqrt{2}} < R < \frac{1}{72}$.

9 a) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Rekka er positiv. I tillegg er $f(x) = \frac{1}{x\sqrt{\ln x}}$

kontinuerlig og avtagende.

Integraltesten:

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \int_{\ln 2}^{\infty} \frac{du}{\sqrt{u}} = \int_{\ln 2}^{\infty} u^{-1/2}\,du$$

$$\left[\begin{array}{c} u = \ln x \\ \frac{du}{dx} = \frac{1}{x} \\ du = \frac{dx}{x} \end{array}\right] \qquad = \left[\, 2u^{1/2} \,\right]_{\ln 2}^{\infty} = \infty.$$

Integralet divergerer, ergo divergerer rekka.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{(n+\sqrt{n})\sqrt{\ln n}}$$

Alternerende rekketest:

$$|a_n| = \frac{1}{(n+\sqrt{n})\sqrt{\ln n}} > \frac{1}{(n+1+\sqrt{n+1})\sqrt{\ln(n+1)}} = |a_{n+1}|$$

$$\lim_{n\to\infty} |a_n| = \frac{1}{(n+\sqrt{n})\sqrt{\ln n}} = 0$$

Rekken konvergerer.

Betinget / absolutt ?

$$\sum_{n=2}^{\infty} \frac{1}{(n+\sqrt{n})\sqrt{\ln n}}$$

Rekka er positiv. Grensesammenligningstesten

$$\lim_{n\to\infty} \frac{\frac{1}{(n+\sqrt{n})\sqrt{\ln n}}}{\frac{1}{n\sqrt{\ln n}}} = \lim_{n\to\infty} \frac{n}{n+\sqrt{n}} = \text{"}\frac{\infty}{\infty}\text{"}$$

$\uparrow$

(fra a)

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{2\sqrt{n}}} = 1$$

Siden $1 > 0$ og $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ er divergent,

må $\sum_{n=2}^{\infty} \frac{1}{(n+\sqrt{n})\sqrt{\ln n}}$ også divergere.

Da må $\sum_{n=2}^{\infty} \frac{(-1)^n}{(n+\sqrt{n})\sqrt{\ln n}}$ konvergere betinget.

b) $$\lim_{n\to\infty}\left|\frac{\frac{2(n+1)+1}{(n+1)!}\;x^{2(n+1)}}{\frac{2n+1}{n!}\;x^{2n}}\right| = \lim_{n\to\infty}|x^2|\cdot\left|\frac{2n+3}{(n+1)!}\cdot\frac{n!}{2n+1}\right|$$

$$= \lim_{n\to\infty}|x^2|\cdot\frac{2n+3}{(n+1)(2n+1)}$$

$$= |x|^2 \lim_{n\to\infty}\frac{2n+3}{(n+1)(2n+1)}$$

$$= |x^2|\cdot 0$$

Vi ser at uansett hvilken x vi velger, vil vi få en verdi som er mindre enn 1.

Så; $x \in \mathbb{R}$ (det vil si at rekka konvergerer for alle x).

$x = 1$: $$\sum_{n=0}^{\infty}\frac{2n+1}{n!} = ?$$

Sett $$\sum_{n=0}^{\infty}\frac{2n+1}{n!}x^{2n} = S(x)$$

$$\sum_{n=0}^{\infty}\frac{1}{n!}x^{2n+1} = \int S(x)\,dx$$

$$x\sum_{n=0}^{\infty}\frac{(x^2)^n}{n!} = \int S(x)\,dx$$

$$xe^{x^2} = \int S(x)\,dx$$

$$\frac{d}{dx}\left(xe^{x^2}\right) = S(x)$$

$$e^{x^2} + xe^{x^2} \cdot 2x = S(x)$$

$$e^{x^2}\left(1 + 2x^2\right) = S(x)$$

$x = 1$: $\quad S(1) = e\,(1+2\cdot 1)$

$$= \underline{\underline{3e}}\,.$$

(Alternativ måte :

$$\sum_{n=0}^{\infty} \frac{2n+1}{n!} = \sum_{n=0}^{\infty} \frac{2n}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$= 2\sum_{n=1}^{\infty} \frac{n}{n!} + e^1$$

$$= 2\sum_{n=1}^{\infty} \frac{1}{(n-1)!} + e$$

$$= 2\sum_{n=0}^{\infty} \frac{1}{n!} + e$$

$$= 2e + e = \underline{\underline{3e}}$$