

Start

Vi starter på nytt

restart

Vi lader inn kommandopakken

with(plots)

```
[animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, (1.1)
conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot,
display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot,
implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot,
listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple,
odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot,
polygonplot3d, polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors,
setoptions, setoptions3d, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d,
tubeplot]
```

Vi lader inn kommandopakken

with(Student[VectorCalculus])

```
[&x, `*`, `+`, `-`, `:`, <, >, <|>, About, ArcLength, BasisFormat, Binormal, ConvertVector,
CrossProduct, Curl, Curvature, D, Del, DirectionalDiff, Divergence, DotProduct,
FlowLine, Flux, GetCoordinates, GetPVDescription, GetRootPoint, GetSpace, Gradient,
Hessian, IsPositionVector, IsRootedVector, IsVectorField, Jacobian, Laplacian, LineInt,
MapToBasis, Nabla, Norm, Normalize, PathInt, PlotPositionVector, PlotVector,
PositionVector, PrincipalNormal, RadiusOfCurvature, RootedVector, ScalarPotential,
SetCoordinates, SpaceCurve, SpaceCurveTutor, SurfaceInt, TNBFrame, Tangent,
TangentLine, TangentPlane, TangentVector, Torsion, Vector, VectorField,
VectorFieldTutor, VectorPotential, VectorSpace, diff, evalVF, int, limit, series]
```

(1.2)

Vi lader inn kommandopakken

with(plottools)

```
[annulus, arc, arrow, circle, cone, cuboid, curve, cutin, cutout, cylinder, disk, dodecahedron,
ellipse, ellipticArc, getdata, hemisphere, hexahedron, homothety, hyperbola, icosahedron,
line, octahedron, parallelepiped, pieslice, point, polygon, prism, project, rectangle, reflect,
rotate, scale, sector, semitorus, sphere, stellate, tetrahedron, torus, transform, translate]
```

(1.3)

Gradienten (Eksamensoppgave 2005 vår / 4)

f er gitt ved

$$f := (x, y) \rightarrow \sin(x) \cdot \cos(y)$$

$$(x, y) \rightarrow \sin(x) \cos(y) \quad (2.1)$$

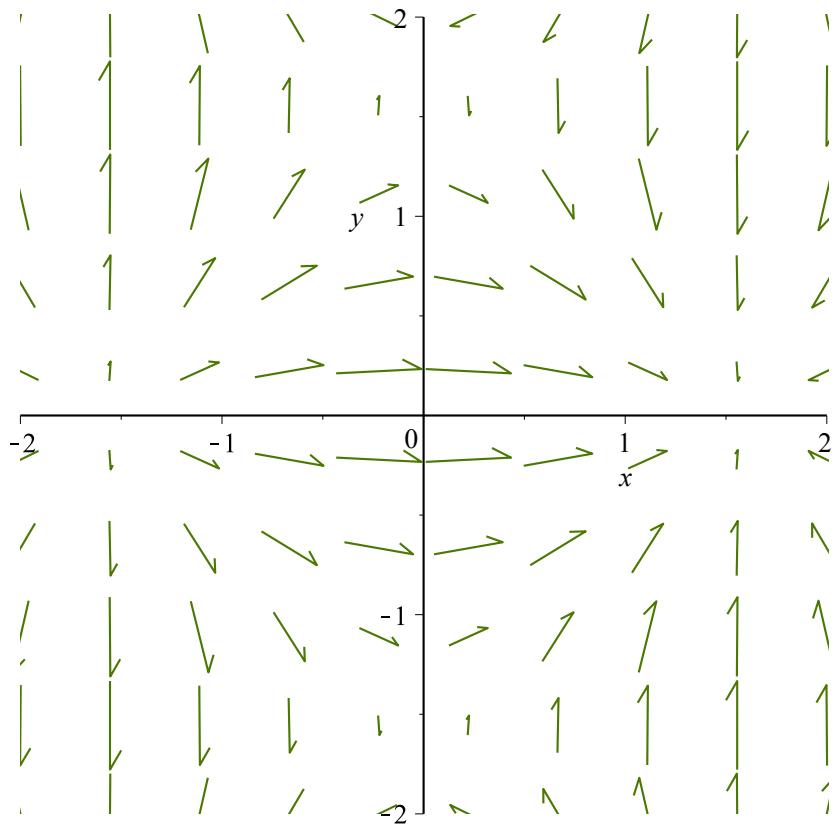
gradienten til f

$$\text{Gradient}(f(x, y))$$

$$(\cos(x) \cos(y)) \bar{e}_x - \sin(x) \sin(y) \bar{e}_y \quad (2.2)$$

gradientfeltet til f

```
VectorField(Gradient(f(x,y)), output=plot, view=[ -2 ..2, -2 ..2], fieldoptions=[grid=[10, 10]])
```



g er gitt ved

$$g := (x, y) \rightarrow y^3 - x^2$$

$$(x, y) \rightarrow y^3 + \text{Student}:-\text{VectorCalculus}:-`(\text{x}^2) \quad (2.3)$$

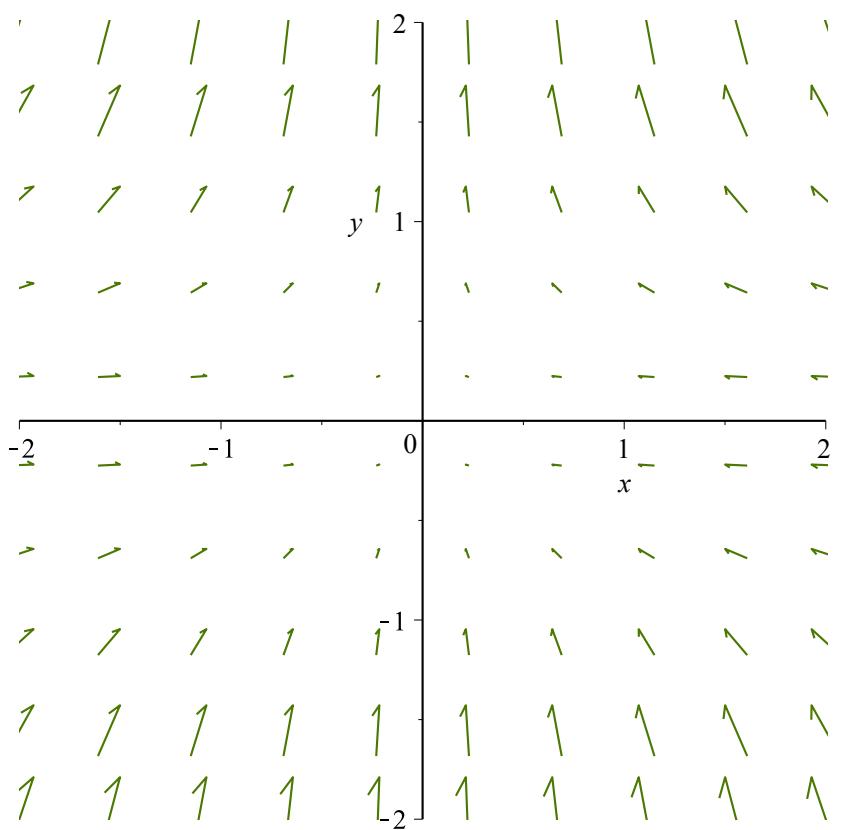
gradienten til g

$$\text{Gradient}(g(x, y))$$

$$-2 \text{x}\bar{e}_x + 3 \text{y}^2\bar{e}_y \quad (2.4)$$

gradientfeltet til g

```
VectorField(Gradient(g(x,y)), output=plot, view=[ -2 ..2, -2 ..2], fieldoptions=[grid=[10, 10]])
```



Kjerneregelen

f er gitt ved (den ytre funksjonen)

$$\mathbf{f} \equiv (x, y) \rightarrow x \cdot y$$

$$(x, y) \rightarrow x y \quad (3.1)$$

x og y er gitt ved (de indre funksjonene)

$$x \equiv t \rightarrow \cos(t)$$

$$t \rightarrow \cos(t) \quad (3.2)$$

$$y \equiv t \rightarrow \sin(t)$$

$$t \rightarrow \sin(t) \quad (3.3)$$

$$\mathbf{diff}(\mathbf{f}(x(t), y(t)), t)$$

$$-\sin(t)^2 + \cos(t)^2 \quad (3.4)$$

Retningsderiverte (Eksamensoppgave 2001/5/b)

Vi skal jobbe med den følgende funksjonen

$$f := (x, y) \rightarrow \begin{cases} \text{piecewise}\left((x^2 + y^2 \leq 400), 20 - \frac{x^2}{25} + \frac{y}{2}, \text{undefined} \right) \\ (x, y) \rightarrow \text{piecewise}\left(x^2 + y^2 \leq 400, 20 + \text{Student:-VectorCalculus:-`-`}\left(1 - \frac{1}{25} x^2 \right) + 1 \frac{1}{2} y, \text{undefined} \right) \end{cases} \quad (4.1)$$

Grafen til f

$$\text{Flaten} := \text{plot3d}(f(x, y), x = -20 .. 20, y = -20 .. 20, \text{axes} = \text{boxed}, \text{numpoints} = 20000, \text{style} = \text{patchcontour})$$

PLOT3D(...) (4.2)

Definisjonsområdet

$$\text{DefOm} := \text{implicitplot}(x^2 + y^2 \leq 400, x = -20 .. 20, y = -20 .. 20, \text{filledregions} = \text{true}, \text{coloring} = ["blue", "gray"], \text{transparency} = 0)$$

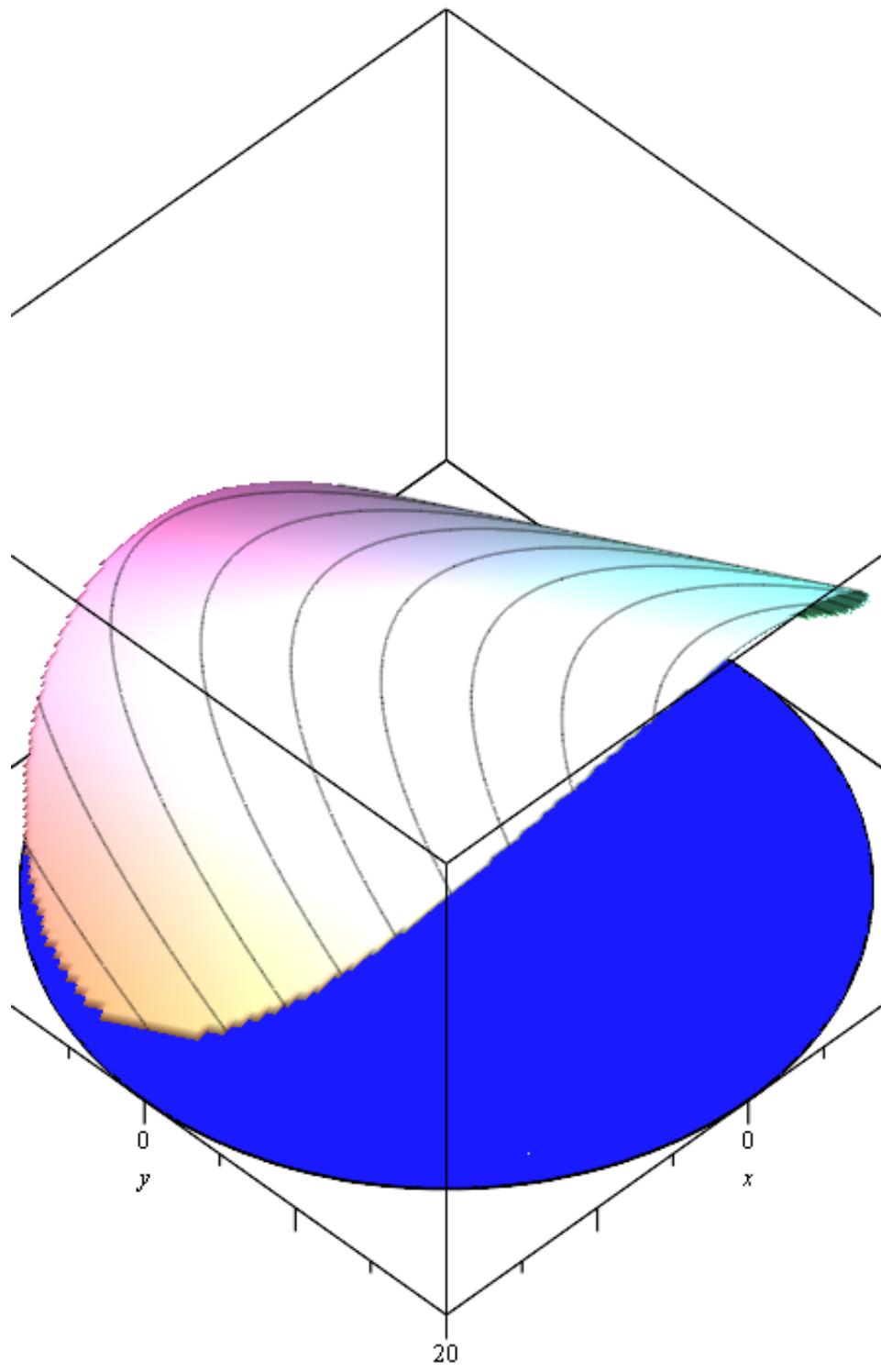
PLOT(...) (4.3)

Konvertering til 3d

$$\text{to3d} := \text{transform}(\ (x, y) \rightarrow [x, y, 0]) :$$

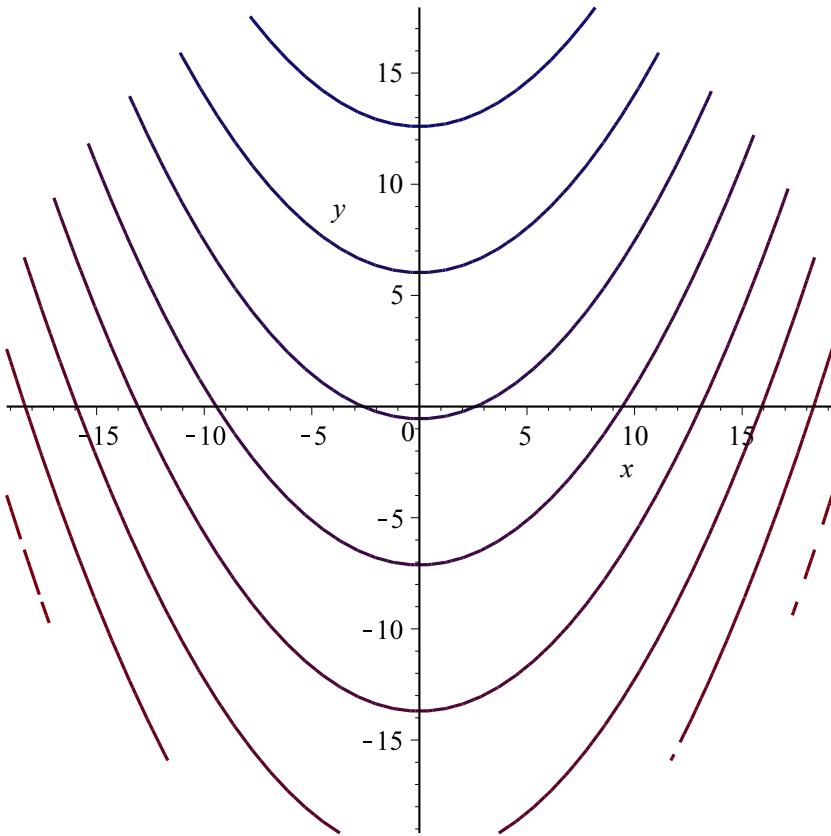
Sammen

$$\text{display}(\text{Flaten}, \text{to3d}(\text{DefOm}), \text{scaling} = \text{constrained})$$



Bilde med nivåkurver

$\text{contourplot}(f(x, y), x = -20 .. 20, y = -20 .. 20);$
 $\text{KONTPLOT} := \text{contourplot}(f(x, y), x = -20 .. 20, y = -20 .. 20)$

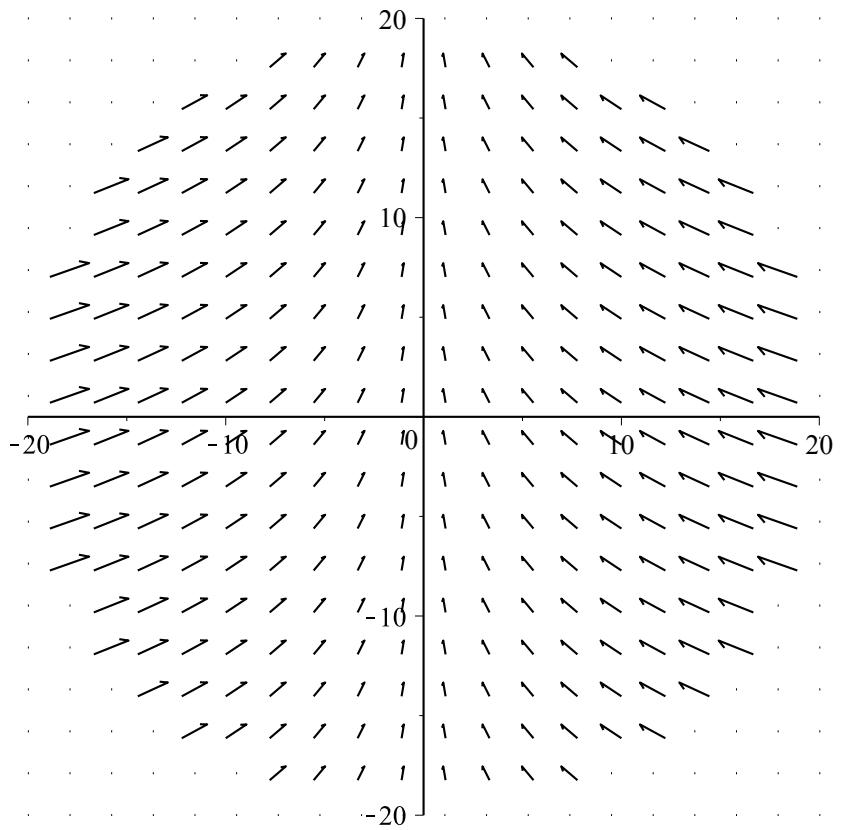


$\text{PLOT}(\dots)$

(4.4)

Bilde med gradienter

$\text{gradplot}(f(x, y), x = -20 .. 20, y = -20 .. 20);$
 $\text{GRADPLOT} := \text{gradplot}(f(x, y), x = -20 .. 20, y = -20 .. 20)$

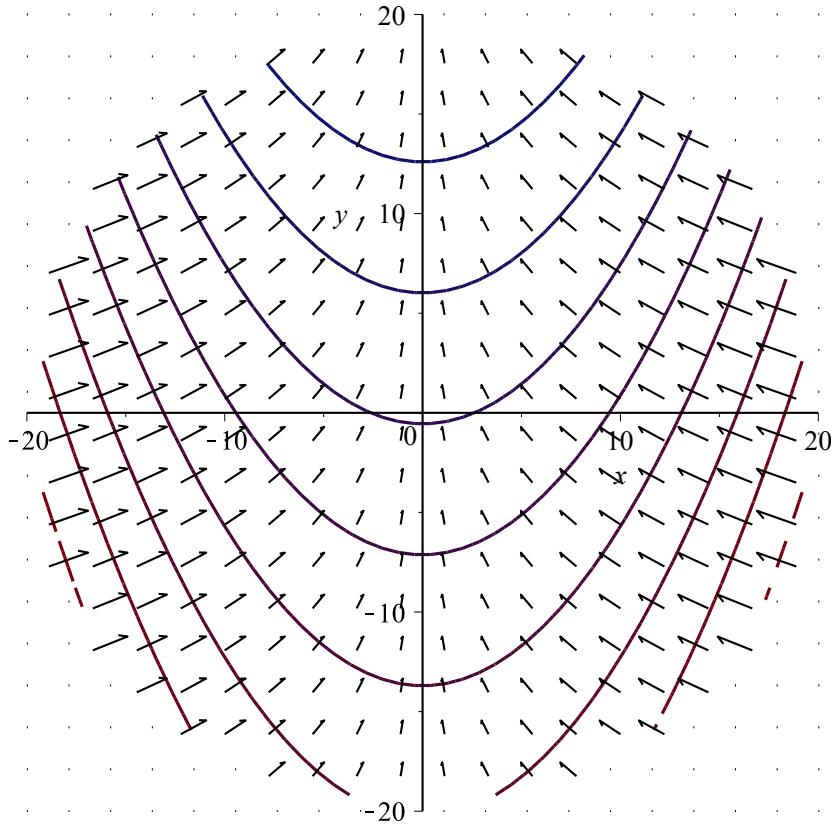


PLOT(...)

(4.5)

Sammen (Merk at gradienten står altid vinkelrett på nivåkurvene)

display(KONTPLOT, GRADPLOT)



Vi definere f over hele planet til å få enklere kommandoer

$$f := (x, y) \rightarrow 20 - \frac{x^2}{25} + \frac{y}{2}$$

$$(x, y) \rightarrow 20 + \text{Student:-VectorCalculus:-`}\left(1 - \frac{1}{25} x^2\right) + 1 \frac{1}{2} y \quad (4.6)$$

Punktet (x_0, y_0)

$$x_0 := 20$$

$$20 \quad (4.7)$$

$$y_0 := 0$$

$$0 \quad (4.8)$$

Punktet (x_0, y_0)

$$\text{Punktet} := \text{sphere}([x_0, y_0, f(x_0, y_0)], 0.05, \text{color} = \text{blue})$$

$$\text{MESH} \left(\begin{array}{c} 1..49 \times 1..49 \times 1..3 \text{ Array} \\ \text{Data Type: } \text{float}_8 \\ \text{Storage: } \text{rectangular} \\ \text{Order: } \text{C_order} \end{array} \right), \text{COLOR}(\text{RGB}, 0., 0., 1.00000000) \quad (4.9)$$

Gradienten her

$$\text{subs}(x = 20, y = 0, \text{Gradient}(f(x, y))) \\ \left(-\frac{8}{5} \right) \bar{e}_x + \left(\frac{1}{2} \right) \bar{e}_y \quad (4.10)$$

$$\text{Grad} := \left\langle -\frac{8}{5}, \frac{1}{2} \right\rangle; \\ \left(-\frac{8}{5} \right) e_x + \left(\frac{1}{2} \right) e_y \quad (4.11)$$

Størrelsen

$\text{Norm}(\text{Grad})$

$$\frac{1}{10} \sqrt{281} \quad (4.12)$$

Gradienten på XY planet

$$\text{GradientVektoren} := \text{plots:-arrow}\left([x0, y0, 0], \left[-\frac{8}{5}, \frac{1}{2}, 0 \right], \text{color} = \text{red}, \text{length} = [1, \text{relative}], \text{width} = 0.3 \right) \\ \text{PLOT3D}(\dots) \quad (4.13)$$

Normalen til skjæringsplanet

$$\text{Normalen} := \text{CrossProduct}\left(\left\langle -\frac{8}{5}, \frac{1}{2}, 0 \right\rangle, \langle 0, 0, 1 \rangle\right) \\ \left(\frac{1}{2} \right) e_x + \left(\frac{8}{5} \right) e_y \quad (4.14)$$

Normalvektoren tegnet i (x0,y0)

$$\text{NormalVektoren} := \text{plots:-arrow}([x0, y0, 0], \text{Normalen}, \text{color} = \text{green}, \text{length} = [1, \text{relative}], \text{width} = 0.3) \\ \text{PLOT3D}(\dots) \quad (4.15)$$

Skjærende planet

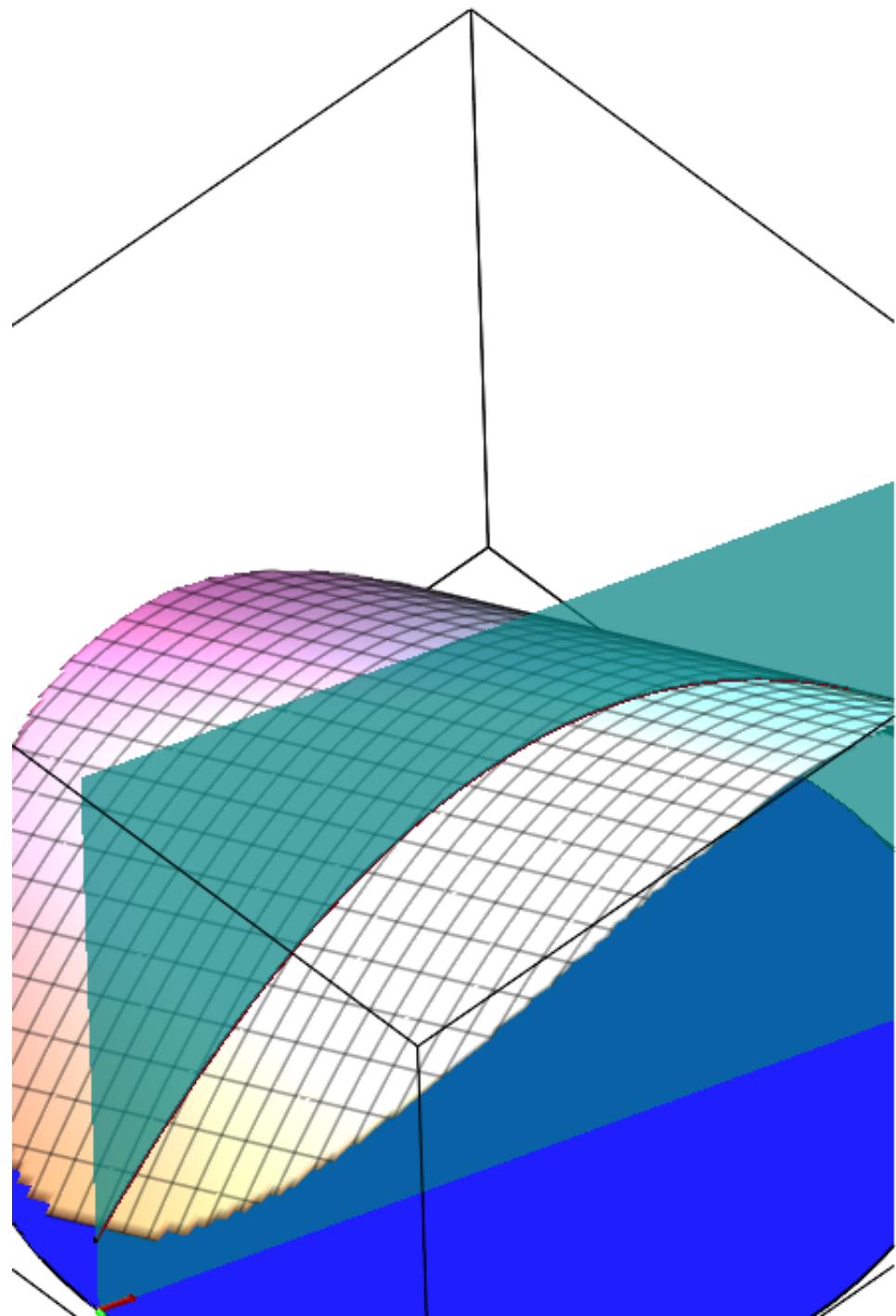
$$\text{Planet} := \text{implicitplot3d}(\text{DotProduct}(\text{Normalen}, (\langle x, y, z \rangle - \langle 20, 0, f(20, 0) \rangle)) = 0, x = -20 .. 20, y = -20 .. 20, z = 0 .. 30, \text{color} = \text{cyan}, \text{style} = \text{surface}, \text{transparency} = 0.3) \\ \text{PLOT3D}(\dots) \quad (4.16)$$

Snittkurven

$$\text{Snittkurven} := \text{SpaceCurve}(\langle x0 + \text{Grad}(1) \cdot t, y0 + \text{Grad}(2) \cdot t, f(x0 + \text{Grad}(1) \cdot t, y0 + \text{Grad}(2) \cdot t) \rangle, t = 0 .. 20, \text{thickness} = 2) \\ \text{PLOT3D}(\dots) \quad (4.17)$$

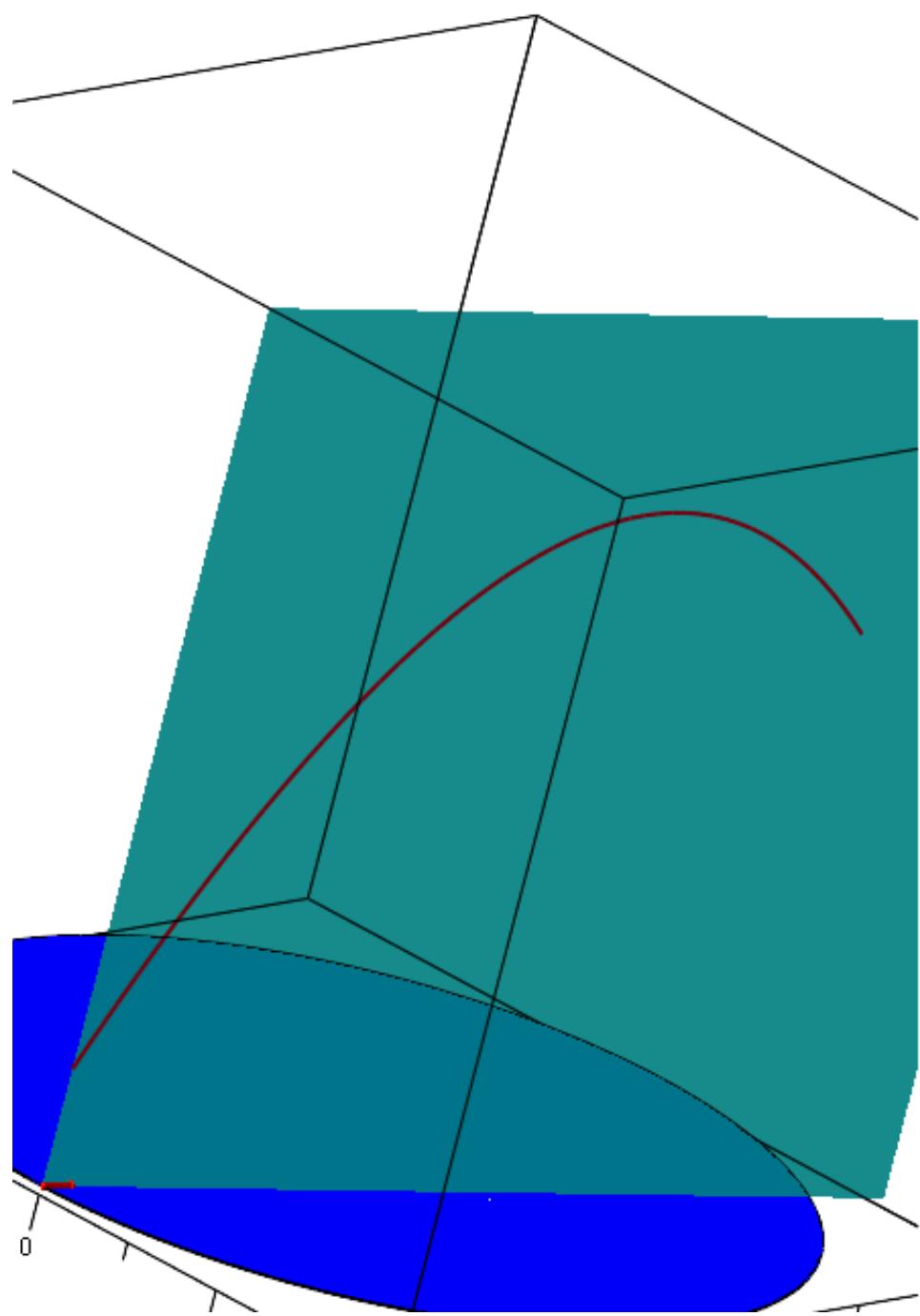
Sammen

$$\text{display}(\text{Flaten}, \text{Punktet}, \text{to3d}(\text{DefOm}), \text{Planet}, \text{GradientVektoren}, \text{NormalVektoren}, \text{Snittkurven}, \text{scaling} = \text{constrained})$$



Uten flaten

display(Planet, Snittkurven, GradientVektoren, to3d(DefOm), axes=boxed)



Retningen av gradienten er

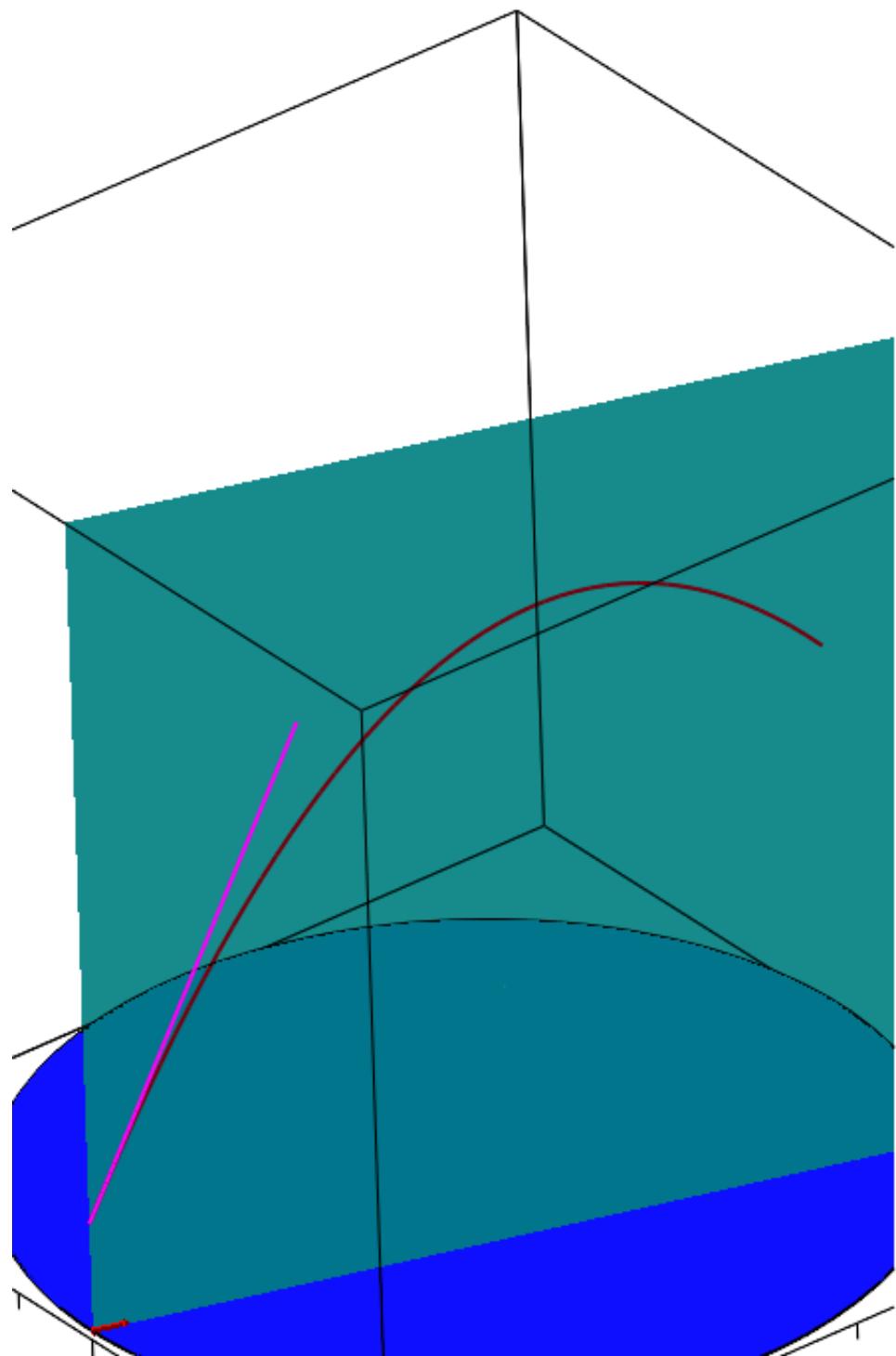
$$\text{GradientRetningen} := \frac{\text{Grad}}{\text{Norm}(\text{Grad})}$$
$$-\frac{16}{281} \sqrt{281} e_x + \frac{5}{281} \sqrt{281} e_y \quad (4.18)$$

Stigningstallet til tangenten til snittkurven er den retningsderiverte

$$\text{TangentLinjen} := \text{SpaceCurve}(\langle x0 + \text{GradientRetningen}(1) \cdot t, y0 + \text{GradientRetningen}(2) \cdot t, \\ f(x0, y0) + \text{Norm}(\text{Grad}) \cdot t \rangle, t = 0 .. 10, \text{color} = \text{magenta}, \text{thickness} = 2)$$
$$\text{PLOT3D}(\dots) \quad (4.19)$$

Uten flaten

`display(Planet, Snittkurven, GradientVektoren, to3d(DefOm), TangentLinjen, axes = boxed)`



L

▼ Tangentplanet, Normallinjen

▼ til nivåflaten til en 3D funksjon f

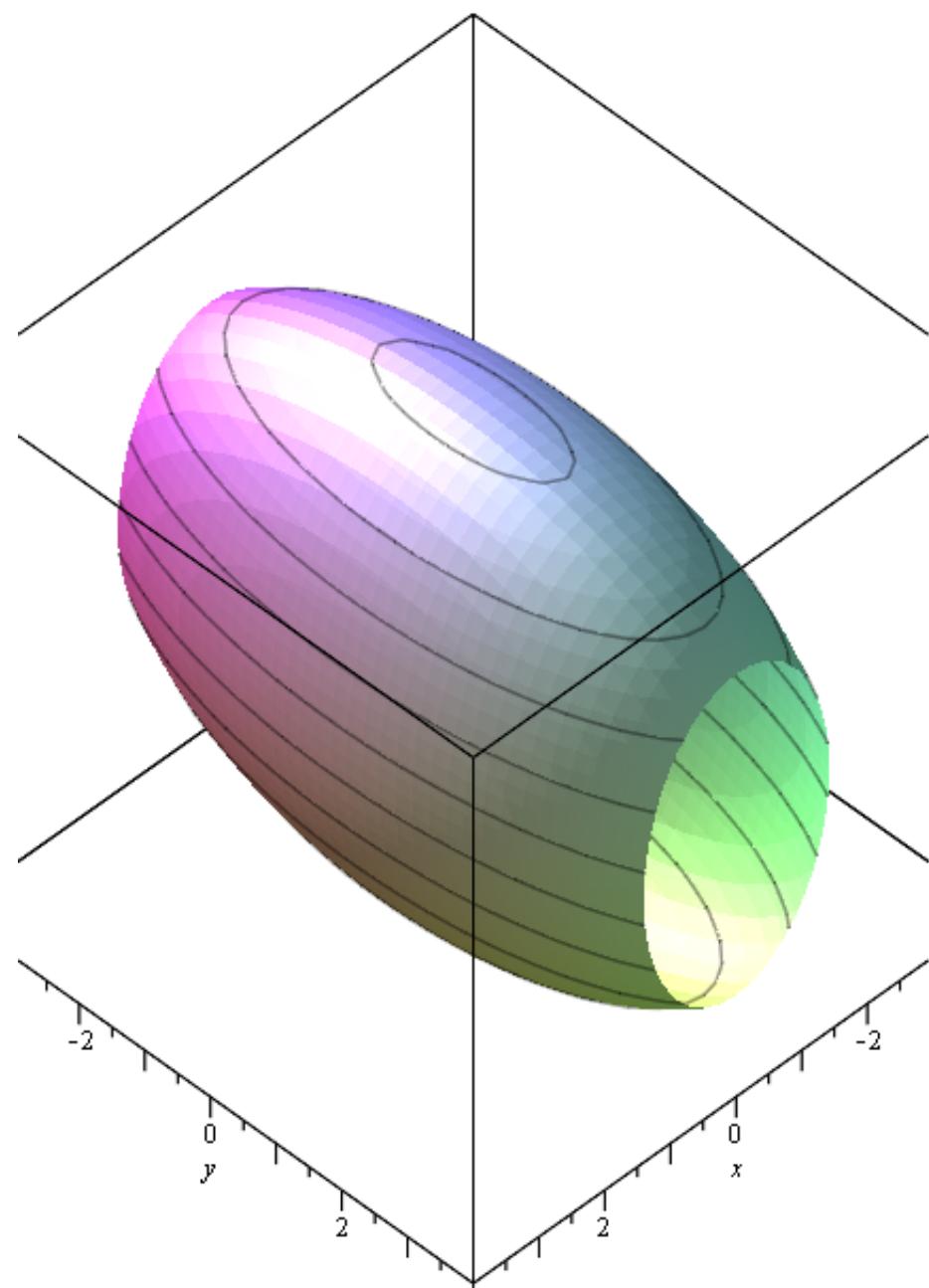
Vi skal jobbe med funksjonen

$$f := (x, y, z) \rightarrow 3 \cdot x^2 + \frac{y^2}{2} + z^2$$

$$(x, y, z) \rightarrow 3x^2 + 1 \frac{1}{2} y^2 + z^2 \quad (5.1.1)$$

Vi tegner nivåflaten tilsv. til $f = 14$

*NivaFlaten := implicitplot3d(f(x, y, z) = 14, x = -4 .. 4, y = -4 .. 4, z = -4 .. 4, grid = [35, 35, 35], axes = boxed, style = patchcontour) :
display(NivaFlaten)*



Punktet P0

$x0 := 1;$

$y0 := 2;$

$z0 := 3;$

1

2

3

(5.1.2)

P0 ligger på nivåflaten $f(x_0, y_0, z_0) = 14$

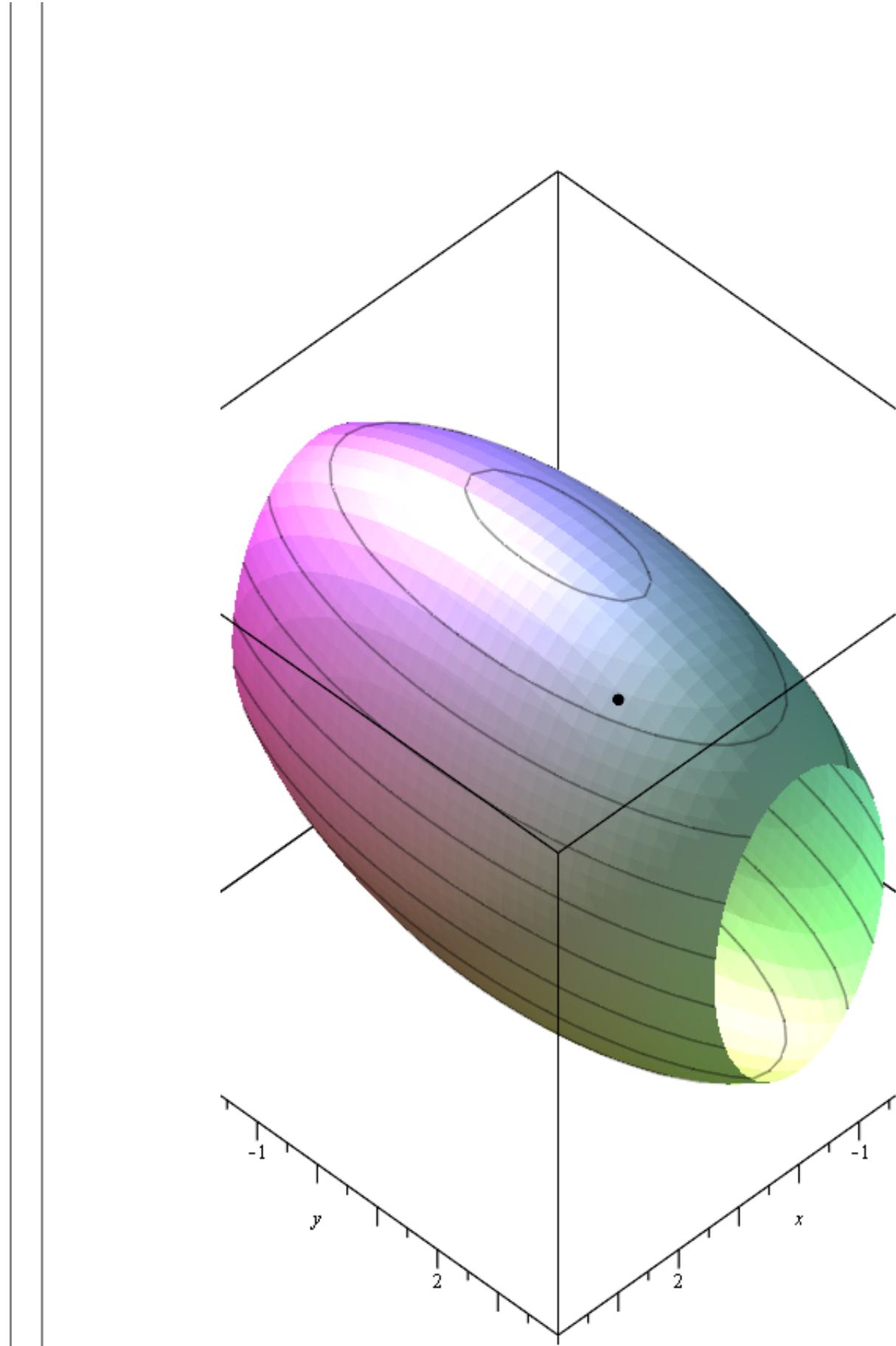
$f(x_0, y_0, z_0)$

14

(5.1.3)

Punktet

*Punktet := sphere([x0, y0, z0], 0.05, color = blue) :
display(NivaFlaten, Punktet)*



Gradienten er

$$\text{Gradient}(f(x, y, z))$$

$$6x\bar{e}_x + (y)\bar{e}_y + 2z\bar{e}_z \quad (5.1.4)$$

Gradienten i P0 er

$$\text{GradP0} := \text{subs}(x = x0, y = y0, z = z0, \text{Gradient}(f(x, y, z)))$$

$$6\bar{e}_x + 2\bar{e}_y + 6\bar{e}_z \quad (5.1.5)$$

Tangentplanet

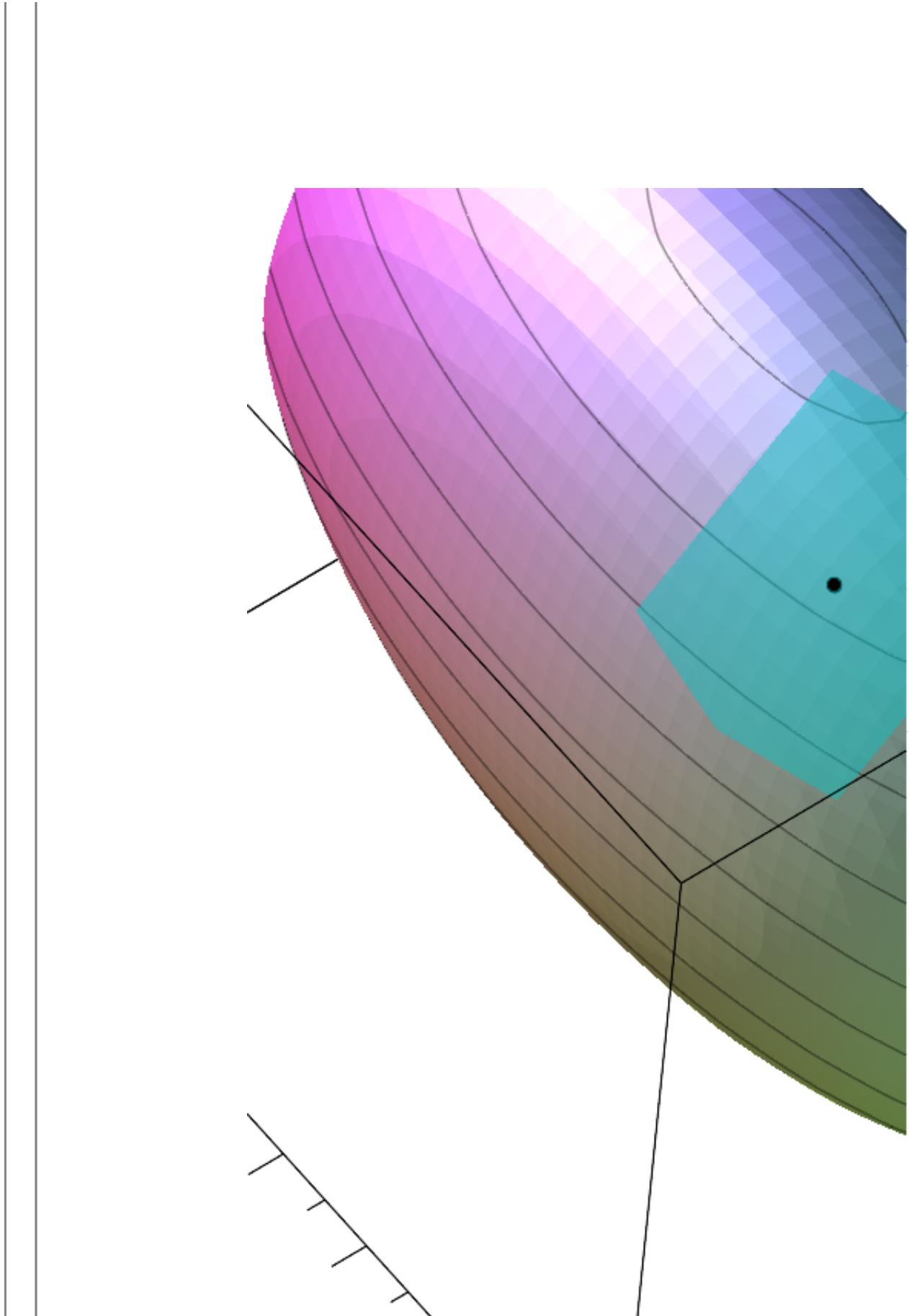
$$\text{TangentPlanet} := \text{implicitplot3d}(\text{GradP0}(1) \cdot (x - x0) + \text{GradP0}(2) \cdot (y - y0) + \text{GradP0}(3)$$

$$\cdot (z - z0) = 0, x = x0 - 1 .. x0 + 1, y = y0 - 1 .. y0 + 1, z = z0 - 1 .. z0 + 1, \text{style} = \text{surface},$$

$$\text{color} = \text{cyan}, \text{transparency} = 0.5);$$

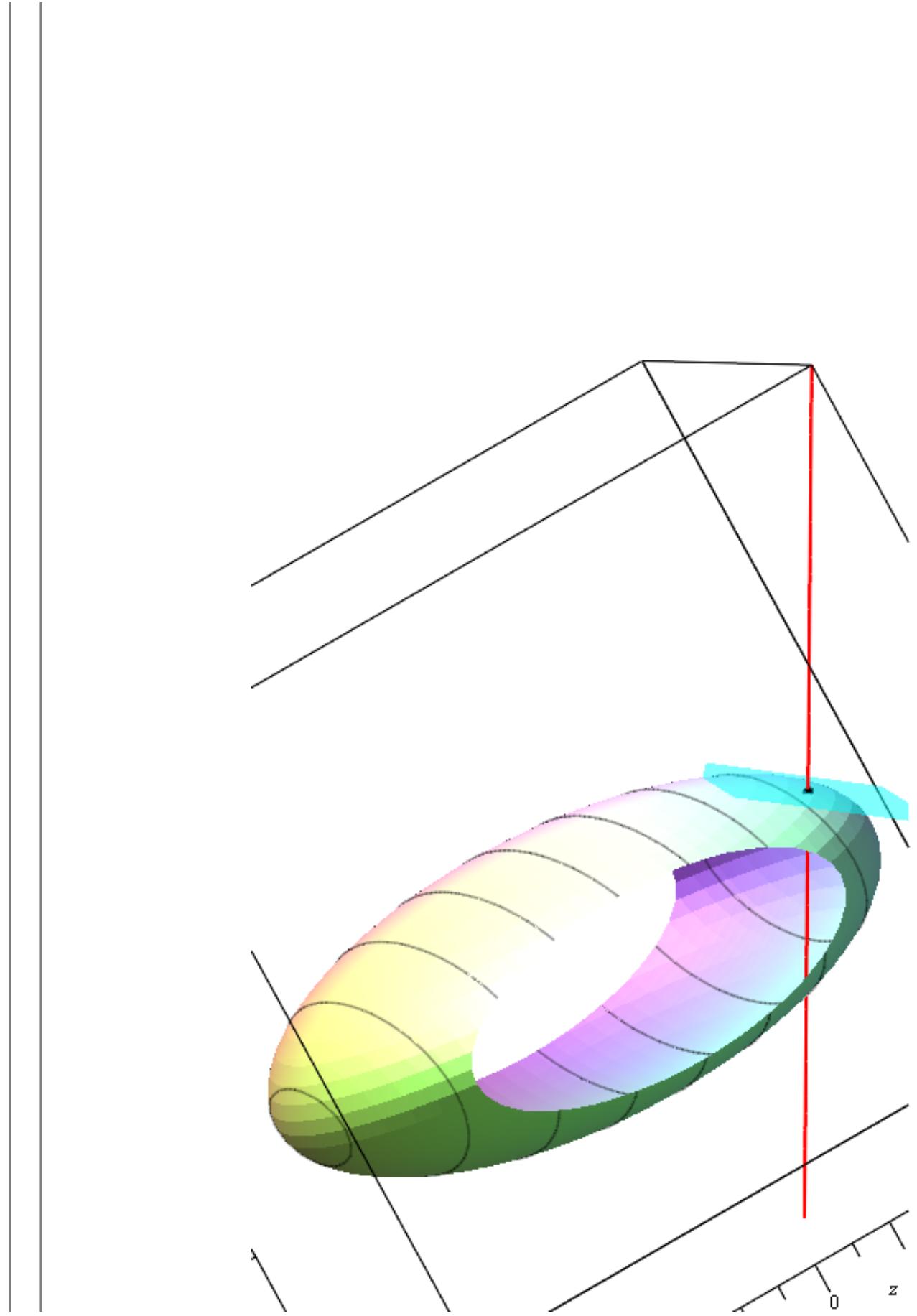
display(NivaFlaten, Punktet, TangentPlanet)

PLOT3D(...)



Normallinjen

```
NormalLinjen := SpaceCurve(⟨x0 + t·GradP0(1), y0 + t·GradP0(2), z0 + t·GradP0(2)⟩, t =  
-1..1, color = red, thickness = 2);  
display(NivaFlaten, Punktet, TangentPlanet, NormalLinjen)  
PLOT3D(...)
```



▼ til grafen til en 2D funksjon f

Vi skal jobbe med denne funksjonen

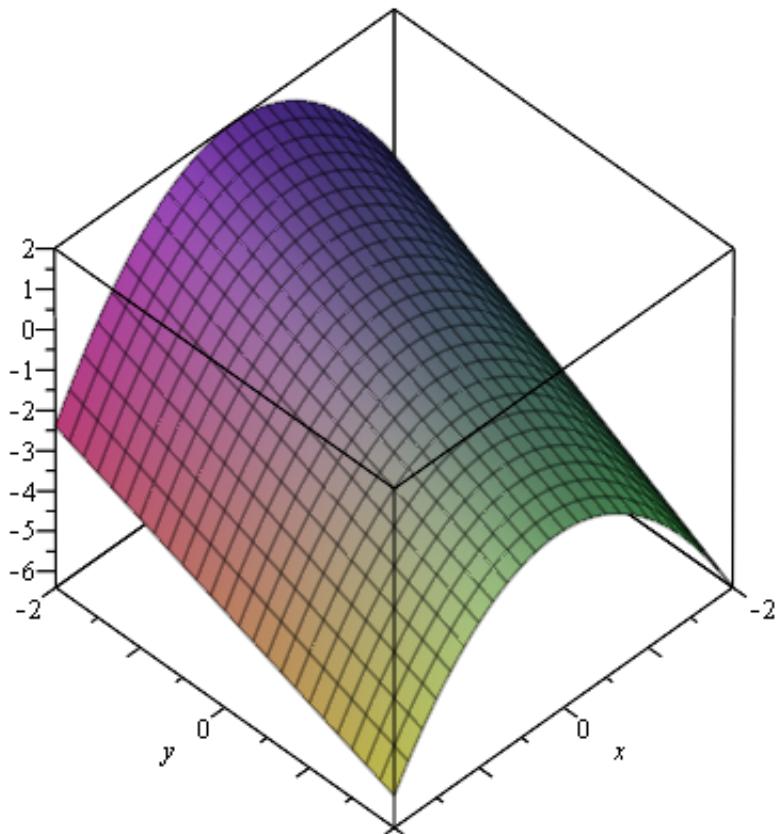
$$f := (x, y) \rightarrow -x^2 - y + \frac{x \cdot y}{10}$$

$$(x, y) \rightarrow \text{Student:-VectorCalculus:-`}(x^2) + \text{Student:-VectorCalculus:-`}(y) + 1 \frac{1}{10} xy \quad (5.2.1)$$

Grafen til f

*Grafen := plot3d(f(x, y), x = -2 .. 2, y = -2 .. 2, axes = boxed);
display(Grafen)*

PLOT3D(...)



Funksjonsverdien i (1,1)

$$f(1, 1)$$

$$-\frac{19}{10}$$

Punktet P0

$$x0 := 1;$$

$$y0 := 1;$$

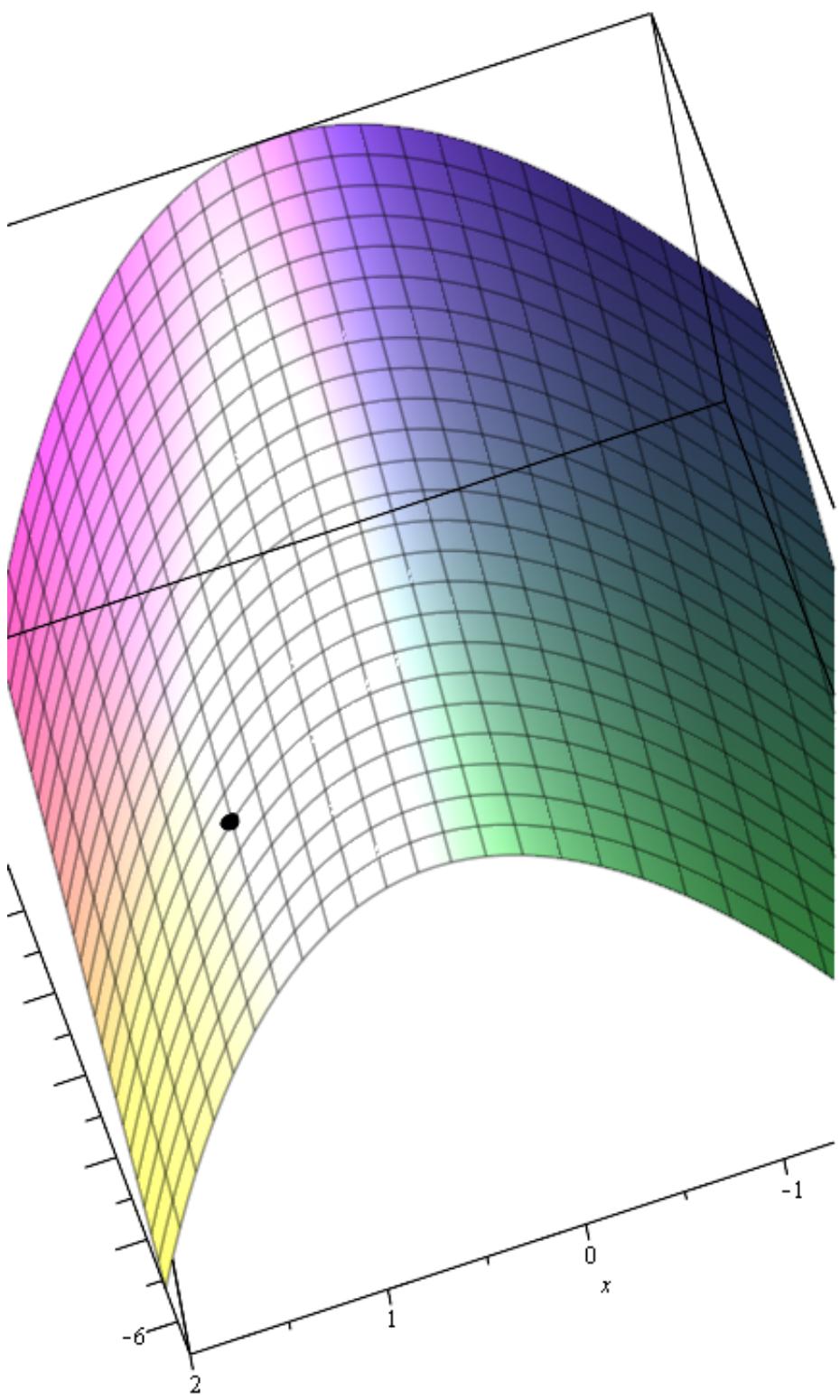
$$\begin{array}{r} z0 := -1.9; \\ \quad \quad \quad 1 \\ \quad \quad \quad 1 \\ \quad \quad \quad -1.9 \end{array} \quad (5.2.3)$$

P0 ligger på grafen: $f(x_0, y_0) = z_0$

$$f(x_0, y_0) = -\frac{19}{10} \quad (5.2.4)$$

Punktet

*Punktet := sphere([x0, y0, z0], 0.05, color = blue) :
display(Grafen, Punktet)*



F er gitt ved

$$F := (x, y, z) \rightarrow (f(x, y) - z)$$
$$(x, y, z) \rightarrow f(x, y) + Student:-VectorCalculus:-`~(z) \quad (5.2.5)$$

Gradienten til F

$$\text{Gradient}(F(x, y, z))$$

$$\left(-2x + \frac{1}{10}y \right) \bar{e}_x + \left(-1 + \frac{1}{10}x \right) \bar{e}_y - \bar{e}_z \quad (5.2.6)$$

Gradienten til F i P0

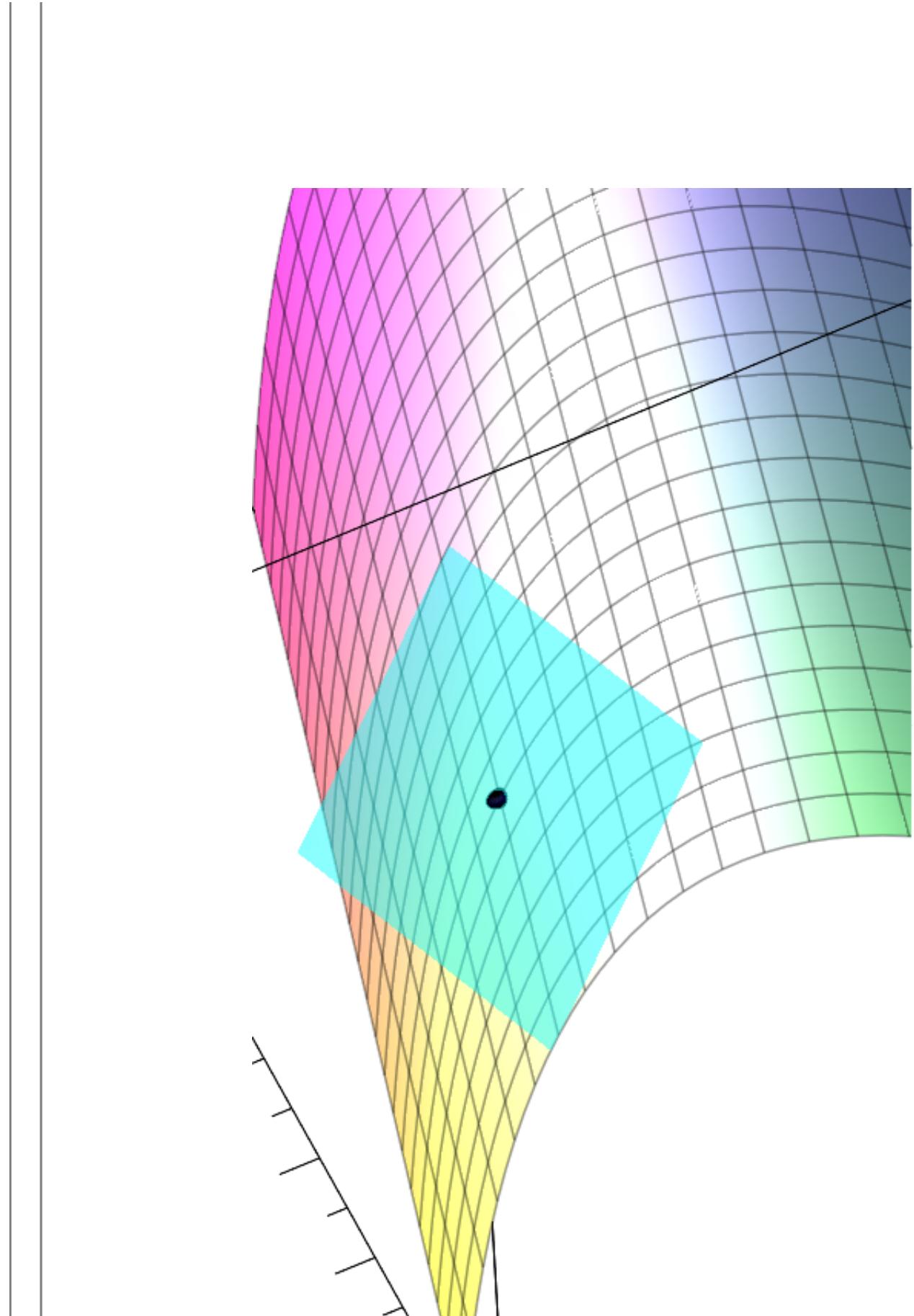
$$GradP0 := subs(x=x0, y=y0, z=z0, \text{Gradient}(F(x, y, z)))$$

$$\left(-\frac{19}{10} \right) \bar{e}_x + \left(-\frac{9}{10} \right) \bar{e}_y - \bar{e}_z \quad (5.2.7)$$

Tangentplanet

$$\begin{aligned} TangentPlanet := & \text{implicitplot3d}(GradP0(1) \cdot (x - x0) + GradP0(2) \cdot (y - y0) + GradP0(3) \\ & \cdot (z - z0) = 0, x = x0 - 1 .. x0 + 1, y = y0 - 1 .. y0 + 1, z = z0 - 1 .. z0 + 1, \text{style=surface}, \\ & \text{color=cyan, transparency=0.5}; \\ & \text{display}(\text{Grafen, Punktet, TangentPlanet}) \end{aligned}$$

$$PLOT3D(...)$$



L L

▼ Lineær Approksimasjon

Vi jobber med funksjonen

$$f := (x, y, z) \rightarrow x^2 \cdot y - x \cdot y \cdot z$$

$$(x, y, z) \rightarrow x^2 y + \text{Student:-VectorCalculus:-`}(x y z)$$

Punktet P0 er

$$P0 := \langle 1, 1, 1 \rangle$$

$$e_x + e_y + e_z \quad (6.2)$$

$$u = \langle 1/2, 1, 1/3 \rangle$$

$$u := \left\langle \frac{1}{2}, 1, \frac{1}{3} \right\rangle;$$

$$\left(\frac{1}{2} \right) e_x + e_y + \left(\frac{1}{3} \right) e_z \quad (6.3)$$

størrelsen av u

$$\text{Norm}(u)$$

$$\frac{7}{6} \quad (6.4)$$

Retningen av u

$$\frac{u}{\text{Norm}(u)}$$

$$\left(\frac{3}{7} \right) e_x + \left(\frac{6}{7} \right) e_y + \left(\frac{2}{7} \right) e_z \quad (6.5)$$

ds

$$ds := \frac{1}{10}$$

$$\frac{1}{10} \quad (6.6)$$

Punktet P ligger i avstand ds fra P0 i retningen av

$$P := P0 + ds \cdot \frac{u}{\text{Norm}(u)}$$

$$\left(\frac{73}{70} \right) e_x + \left(\frac{38}{35} \right) e_y + \left(\frac{36}{35} \right) e_z \quad (6.7)$$

Funksjonsverdien i P0

$$f(P0(1), P0(2), P0(3))$$

$$0 \quad (6.8)$$

Gradienten til f

$$\text{Gradient}(f(x, y, z))$$

$$(2xy - yz)\bar{e}_x + (x^2 - xz)\bar{e}_y - xy\bar{e}_z \quad (6.9)$$

Gradienten til f i P0

$$\text{GradP0} := \text{subs}(x = P0(1), y = P0(2), z = P0(3), \text{Gradient}(f(x, y, z)))$$

$$\bar{e}_x - \bar{e}_z \quad (6.10)$$

df

$$df := \text{DotProduct}\left(\text{GradP0}, \frac{u}{\text{Norm}(u)}\right) \cdot ds \quad \frac{1}{70} \quad (6.11)$$

Vår approksimasjon

$$\text{evalf}(f(P0(1), P0(2), P0(3)) + df) \quad 0.01428571429 \quad (6.12)$$

Funksjonsverdien i P

$$\text{evalf}(f(P(1), P(2), P(3))) \quad 0.01617492711 \quad (6.13)$$

Bytt til $ds := 1/100$ og se hva skjer!