Start

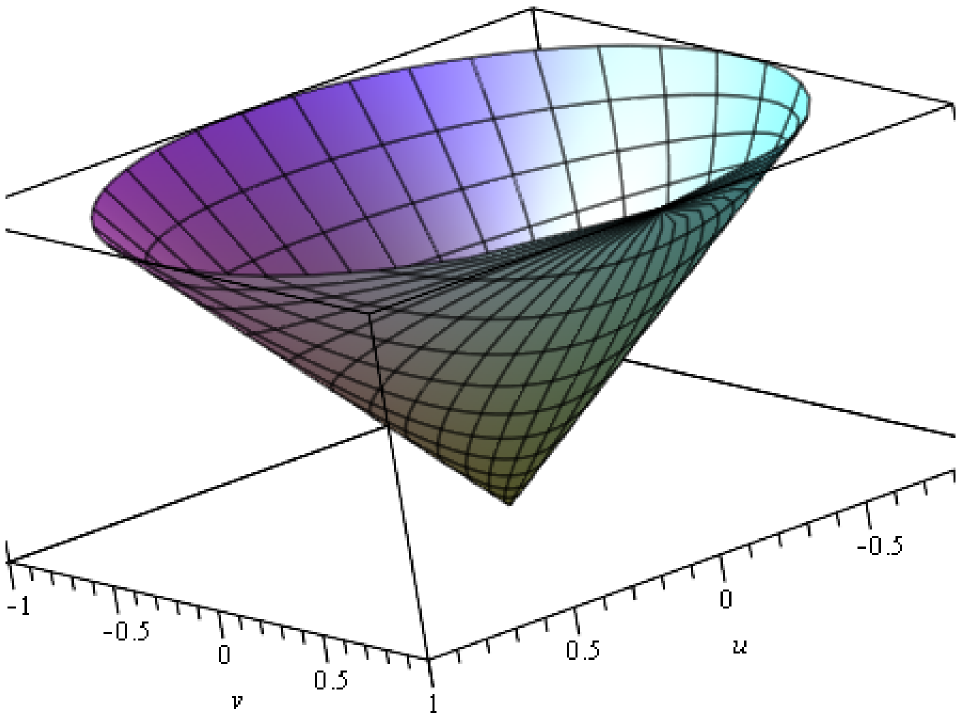
Eksempel 1

Finn treghetsmomenten om z-aksen til den parametriske flaten <2uv, u^2 - v^2, u^2 + v^2> med u^2+v^2 <= 1 og massetetheten konstant 1

$flate := plot3d\left(\left[2 \cdot u \cdot v, u^2 - v^2, u^2 + v^2\right], u = -1 ..1, v = -\mathrm{sqrt}\left(1 - u^2\right) ..\mathrm{sqrt}\left(1 - u^2\right), axes = boxed, scaling = constrained\right);$

$display(flate);$

$$PLOT3D(\ldots)$$



Flaten er en elliptisk kjegle: x^2 + y^2 = z^2

Parameterfremstillingen

$r := (u, v) \rightarrow \langle 2 \cdot u \cdot v, u^2 - v^2, u^2 + v^2 \rangle;$

$(u, v) \rightarrow Student\text{:-}VectorCalculus\text{:-} <,> (2\, u\, v, u^2 + Student\text{:-}VectorCalculus\text{:-}`\text{-}`(v^2), u^2$ **(2.1)**

$+v^2)$

Tangentvektorer (partiell deriverte)

ru := *diff*(*r*(*u*, *v*), *u*);

$$2\,v e_x + 2\,u e_y + 2\,u e_z \tag{2.2}$$

rv := *diff*(*r*(*u*, *v*), *v*);

$$2\,u e_x - 2\,v e_y + 2\,v e_z \tag{2.3}$$

r_u x r_v

CrossProduct(*ru*, *rv*);

$$8\,u\,v e_x + (4\,u^2 - 4\,v^2) e_y + (-4\,v^2 - 4\,u^2) e_z \tag{2.4}$$

| r_u x r_v |

Norm(*CrossProduct*(*ru*, *rv*));

$$4\sqrt{2}\,(u^2 + v^2) \tag{2.5}$$

Treghetsmomenten

$Int\left((u^2+v^2)^2 \cdot 4 \cdot \sqrt{2} \cdot (u^2+v^2),\ v = -\mathrm{sqrt}(1-u^2)\ ..\mathrm{sqrt}(1-u^2),\ u = -1\ ..1\right);$

$$\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} 4\,(u^2+v^2)^3\sqrt{2}\,\mathrm{d}v\,\mathrm{d}u \tag{2.6}$$

Treghetsmomenten ved polar koordinater

$MultiInt\left(4\cdot\sqrt{2}\cdot r^6\cdot r,\ r = 0\ ..1,\ \mathrm{theta} = 0\ ..2\cdot\mathrm{Pi},\ output = steps\right);$

$$\int_0^{2\pi}\int_0^1 4\sqrt{2}\,r^7\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{2\pi}\left(\left.\frac{\sqrt{2}\,r^8}{2}\right|_{r=0\,..1}\right)\mathrm{d}\theta$$

$$= \int_0^{2\pi}\frac{\sqrt{2}}{2}\,\mathrm{d}\theta$$

$$= \left.\frac{\sqrt{2}\,\theta}{2}\right|_{\theta=0\,..2\pi}$$

$$\sqrt{2}\,\pi \tag{2.7}$$

Eksempel 2

Finn momenten om xy-planet til S som er gitt ved z^2 = 1 + x^2 + y^2 og ligger mellom planene z=1 og z=$\sqrt{5}$ og har massetethet konstant 1

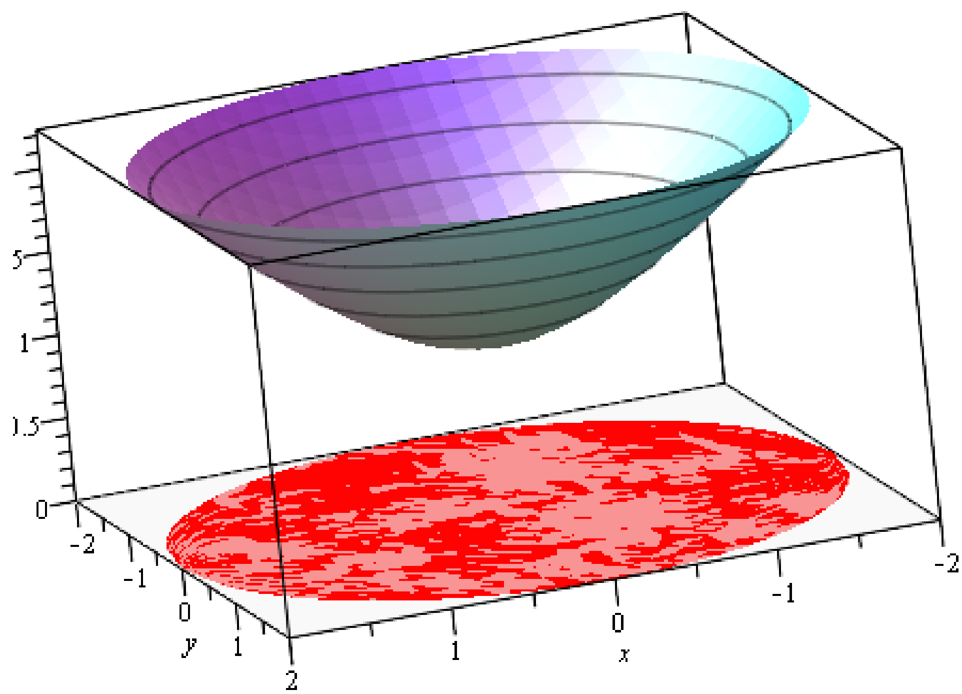
$flate := implicitplot3d\left(z^2 = 1 + x^2 + y^2,\ x = -2\ ..2,\ y = -2\ ..2,\ z = 1\ ..\sqrt{5},\ style = surfacecontour,\ grid\right.$

$= [25\ ,25, 25]\big)$:

$XYplanet := implicitplot3d(z = 0, x = -2\ ..2, y = -2\ ..2, z = 0\ ..1, color = gray, style = surface, transparency = 0.4)$:

$Skyggen := plot3d\big([x, y, 0], x = -2\ ..2, y = -\mathrm{sqrt}\big(4 - x^2\big)\ ..\mathrm{sqrt}\big(4 - x^2\big), color = red, style = surface\big)$:

$display(flate, XYplanet, Skyggen, axes = boxed, scaling = constrained)$;



Flaten er en elliptisk paraboloide gitt ved f(x,y,z) = 0 (en nivåflate til f !!!)

$f := (x, y, z) \rightarrow x^2 + y^2 - z^2 + 1;$

$$(x, y, z) \rightarrow x^2 + y^2 + \mathit{Student{:}\text{-}VectorCalculus{:}\text{-}`\text{-}`}\left(z^2\right) + 1 \tag{3.1}$$

Gradienten til f og k komponenten

$Gradient(f(x, y, z), [x, y, z]);$

$$\begin{bmatrix} 2\,x \\ 2\,y \\ -2\,z \end{bmatrix} \tag{3.2}$$

$Norm(Gradient(f(x, y, z), [x, y, z]));$

$$2\sqrt{x^2 + y^2 + z^2} \tag{3.3}$$

$DotProduct(Gradient(f(x, y, z), [x, y, z]), \langle 0, 0, 1\rangle);$

$$-2\,z \tag{3.4}$$

$|DotProduct(Gradient(f(x, y, z), [x, y, z]), \langle 0, 0, 1\rangle)|;$

$$2\,|z| \tag{3.5}$$

Momenten (z^2 = x^2 + y^2 + 1 skrives inn)

$Int\left(z \cdot \frac{2\cdot\sqrt{x^2+y^2+z^2}}{2\cdot z}, y = -\mathrm{sqrt}(4-x^2)\,..\,\mathrm{sqrt}(4-x^2), x = -1\,..\,1\right)$

$= Int\left(\sqrt{x^2+y^2+x^2+y^2+1}, y = -\mathrm{sqrt}(4-x^2)\,..\,\mathrm{sqrt}(4-x^2), x = -1\,..\,1\right);$

$$\int_{-1}^{1}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+y^2+z^2}\,\mathrm{d}y\,\mathrm{d}x = \int_{-1}^{1}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{2\,x^2+2\,y^2+1}\,\mathrm{d}y\,\mathrm{d}x \tag{3.6}$$

Momenten ved polar koordinater

$MultiInt\left(\mathrm{sqrt}(1+2\cdot r^2)\cdot r, r = 0\,..\,2, \mathrm{theta} = 0\,..\,2\cdot\mathrm{Pi}, output = steps\right);$

$$\int_0^{2\pi}\int_0^2 \sqrt{1+2\,r^2}\;r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{2\pi} \left(\left.\frac{\left(1+2\,r^2\right)^{3/2}}{6}\right|_{r=0\,..\,2} \right) \mathrm{d}\theta$$

$$= \int_0^{2\pi} \frac{13}{3}\,\mathrm{d}\theta$$

$$= \left.\frac{13\,\theta}{3}\right|_{\theta=0\,..\,2\,\pi}$$

$$\frac{26}{3}\,\pi \tag{3.7}$$

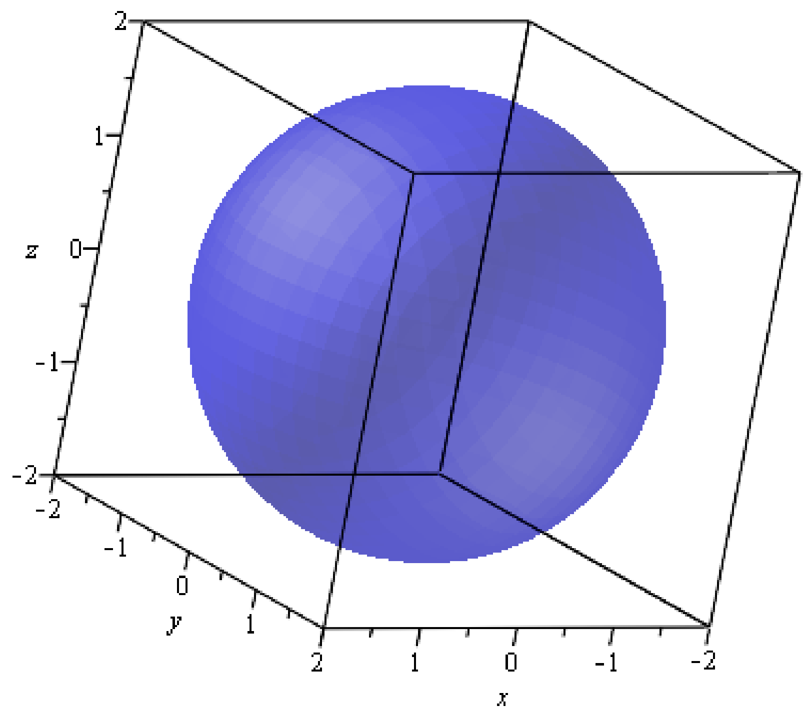
Eksempel 3

Finn fluksen til F=m*r/|r|^3 ut fra en sfære med radius a og senteret i origo

$flate := implicitplot3d\left(x^2+y^2+z^2=4, x=-2..2, y=-2..2, z=-2..2, style=surface, scaling=constrained, axes=boxed, transparency=0.6, color=blue, grid=[20, 20, 20]\right);$

$display(flate);$

$$PLOT3D(\ldots)$$

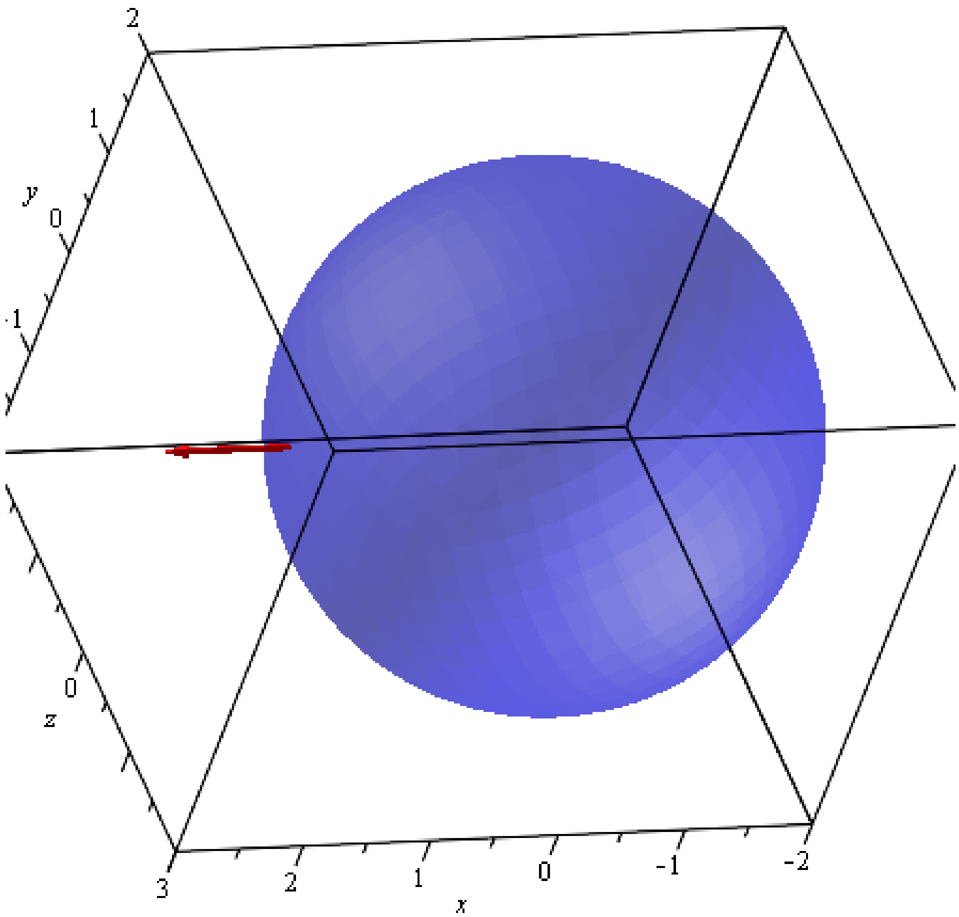


Normalvektoren

$n := (x, y, z) \rightarrow \frac{\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}};$

$$(x, y, z) \rightarrow Student:\text{-}VectorCalculus:\text{-}<,>(x, y, z)\ \frac{1}{\sqrt{x^2+y^2+z^2}} \qquad \textbf{(4.1)}$$

$display(flate, arrow(\langle 2, 0, 0\rangle, n(2, 0, 0), color=red));$



Fluksen

$$\iint_S \frac{m\cdot\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}^3}\cdot\frac{\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}}\,\mathrm{d}\sigma = \iint_S \frac{m}{\sqrt{x^2+y^2+z^2}^2}\cdot\frac{\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}}\cdot\frac{\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}}\,\mathrm{d}\sigma$$

$$=\iint_S \frac{m}{\sqrt{x^2+y^2+z^2}^{\,2}} \cdot d\sigma$$

Fluks ved kulekoordinater

$MultiInt\left(\frac{m}{a^2} \cdot a^2 \cdot \sin(\phi), \phi = 0..\text{Pi}, \theta = 0..2 \cdot \text{Pi}, output = steps\right);$

$$\int_0^{2\pi} \int_0^{\pi} m \sin(\phi)\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \left(-m \cos(\phi) \Big|_{\phi = 0..\pi} \right) d\theta$$

$$= \int_0^{2\pi} 2\, m\, d\theta$$

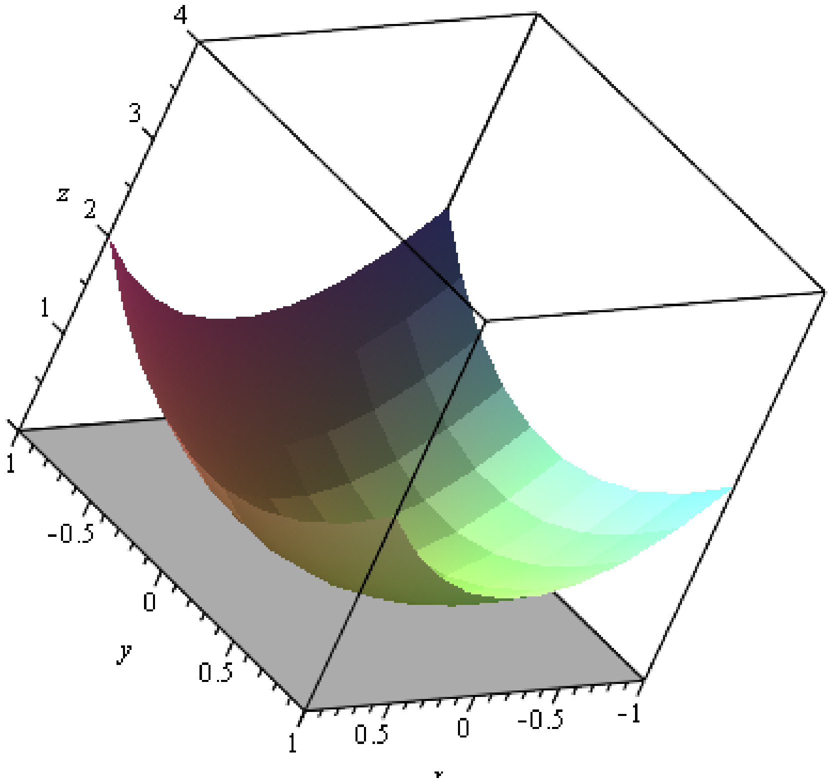
$$= 2\, m\, \theta \Big|_{\theta = 0..2\pi}$$

$$4\, m\, \pi \tag{4.2}$$

Eksempel 6

Finn fluksen til F=<z,0,x^2> gjennom flaten S gitt ved z = x^2 + y^2 med den følgende skyggen på xy-planet:
R = {(x,y) : -1 <= x <= 1 og -1 <= y <= 1}

$flate := implicitplot3d\left(z = x^2 + y^2, x = -1..1, y = -1..1, z = 0..4, axes = boxed, style = surface\right):$
$XYplanet := implicitplot3d(z = 0, x = -1..1, y = -1..1, z = 0..1, color = gray, style = surface, transparency = 0.4):$
$display(flate, XYplanet);$



Normalfeltet

$f := (x, y, z) \rightarrow z - x^2 - y^2;$

$$(x, y, z) \rightarrow z + \mathit{Student}{:}\text{-}\mathit{VectorCalculus}{:}\text{-}\grave{}\text{-}\grave{}\left(x^2\right) + \mathit{Student}{:}\text{-}\mathit{VectorCalculus}{:}\text{-}\grave{}\text{-}\grave{}\left(y^2\right) \tag{5.1}$$

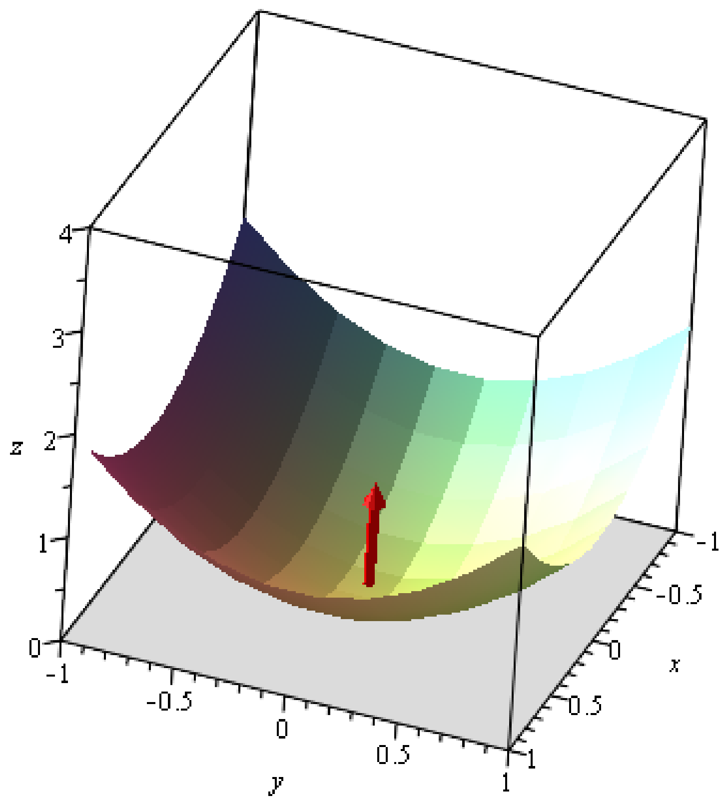
$\mathit{Gradient}(f(x, y, z), [x, y, z]);$

$$\begin{bmatrix} -2\,x \\ -2\,y \\ 1 \end{bmatrix} \tag{5.2}$$

$n := (x, y, z) \rightarrow \dfrac{\langle -2 \cdot x, -2 \cdot y, 1 \rangle}{\mathit{DotProduct}(\langle -2 \cdot x, -2 \cdot y, 1 \rangle, \langle 0, 0, 1 \rangle)};$

$$\begin{aligned} &(x, y, z) \rightarrow \mathit{Student}{:}\text{-}\mathit{VectorCalculus}{:}\text{-}{<,>}(\mathit{Student}{:}\text{-}\mathit{VectorCalculus}{:}\text{-}\grave{}\text{-}\grave{}(2\,x), \\ &\quad \mathit{Student}{:}\text{-}\mathit{VectorCalculus}{:}\text{-}\grave{}\text{-}\grave{}(2\,y), 1)\ 1 \big/ (\mathit{Student}{:}\text{-}\mathit{VectorCalculus}{:}\text{-} \\ &\quad \mathit{DotProduct}(\mathit{Student}{:}\text{-}\mathit{VectorCalculus}{:}\text{-}{<,>}(\mathit{Student}{:}\text{-}\mathit{VectorCalculus}{:}\text{-}\grave{}\text{-}\grave{}(2\,x), \\ &\quad \mathit{Student}{:}\text{-}\mathit{VectorCalculus}{:}\text{-}\grave{}\text{-}\grave{}(2\,y), 1), \mathit{Student}{:}\text{-}\mathit{VectorCalculus}{:}\text{-}{<,>}(0, 0, 1))) \end{aligned} \tag{5.3}$$

$\mathit{display}(\mathit{flate}, \mathit{XYplanet}, \mathit{arrow}(n(0, 0, 0), \mathit{color} = \mathit{red}));$



Fluksen

$MultiInt(x^2 - 2\cdot x^3 - 2\cdot x\cdot y^2, x = -1\,..1, y = -1\,..1, output = steps);$

$$\int_{-1}^{1}\int_{-1}^{1} \left(x^2 - 2\,x^3 - 2\,x\,y^2\right)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{-1}^{1} \left(\left(\frac{1}{3}\,x^3 - \frac{1}{2}\,x^4 - x^2\,y^2 \right)\Bigg|_{x=-1\,..1} \right) \mathrm{d}y$$

$$= \int_{-1}^{1} \frac{2}{3}\,\mathrm{d}y$$

$$= \frac{2\,y}{3}\Bigg|_{y=-1\,..1}$$

$$\frac{4}{3} \tag{5.4}$$