Start

restart;
with(*Student*[*MultivariateCalculus*]);

[*ApproximateInt*, *ApproximateIntTutor*, *CenterOfMass*, *ChangeOfVariables*, *CrossSection*, *CrossSectionTutor*, *Del*, *DirectionalDerivative*, *DirectionalDerivativeTutor*, *FunctionAverage*, *Gradient*, *GradientTutor*, *Jacobian*, *LagrangeMultipliers*, *MultiInt*, *Nabla*, *Revert*, *SecondDerivativeTest*, *SurfaceArea*, *TaylorApproximation*, *TaylorApproximationTutor*] **(1.1)**

with(*plots*);

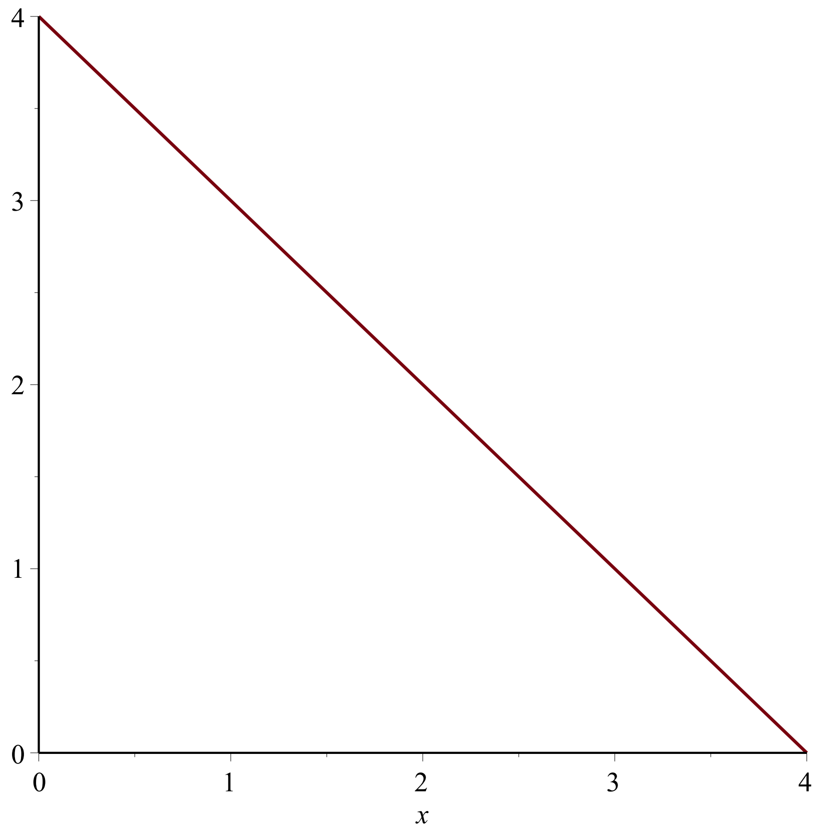
[*animate*, *animate3d*, *animatecurve*, *arrow*, *changecoords*, *complexplot*, *complexplot3d*, *conformal*, *conformal3d*, *contourplot*, *contourplot3d*, *coordplot*, *coordplot3d*, *densityplot*, *display*, *dualaxisplot*, *fieldplot*, *fieldplot3d*, *gradplot*, *gradplot3d*, *implicitplot*, *implicitplot3d*, *inequal*, *interactive*, *interactiveparams*, *intersectplot*, *listcontplot*, *listcontplot3d*, *listdensityplot*, *listplot*, *listplot3d*, *loglogplot*, *logplot*, *matrixplot*, *multiple*, *odeplot*, *pareto*, *plotcompare*, *pointplot*, *pointplot3d*, *polarplot*, *polygonplot*, *polygonplot3d*, *polyhedra_supported*, *polyhedraplot*, *rootlocus*, *semilogplot*, *setcolors*, *setoptions*, *setoptions3d*, *spacecurve*, *sparsematrixplot*, *surfdata*, *textplot*, *textplot3d*, *tubeplot*] **(1.2)**

Eksempel 1

Finn $\int\int e^{x-y} dA$ over R

R er området

plot$(4 - x, x = 0 ..4)$;



dydx:
$MultiInt(\exp(x-y), y=0..4-x, x=0..4, output=steps);$

$$\int_0^4\int_0^{4-x} e^{x-y}\,dy\,dx$$

$$= \int_0^4 \left(-e^{x-y}\Big|_{y=0..4-x} \right) dx$$

$$= \int_0^4 \left(e^x - e^{2x-4} \right) dx$$

$$= \left(e^x - \frac{e^{2x-4}}{2} \right)\Bigg|_{x=0..4}$$

$$-1 + \frac{1}{2}e^{-4} + \frac{1}{2}e^4 \qquad \textbf{(2.1)}$$

dxdy:
$MultiInt(\exp(x-y), x=0..4-y, y=0..4, output=steps);$

$$\int_0^4 \int_0^{4-y} e^{x-y}\,dx\,dy$$

$$= \int_0^4 \left(e^{x-y} \Big|_{x=0..4-y} \right) dy$$

$$= \int_0^4 \left(-e^{-y} + e^{4-2y} \right) dy$$

$$= \left(e^{-y} - \frac{e^{4-2y}}{2} \right) \Bigg|_{y=0..4}$$

$$-1 + \frac{1}{2} e^{-4} + \frac{1}{2} e^{4} \qquad \textbf{(2.2)}$$

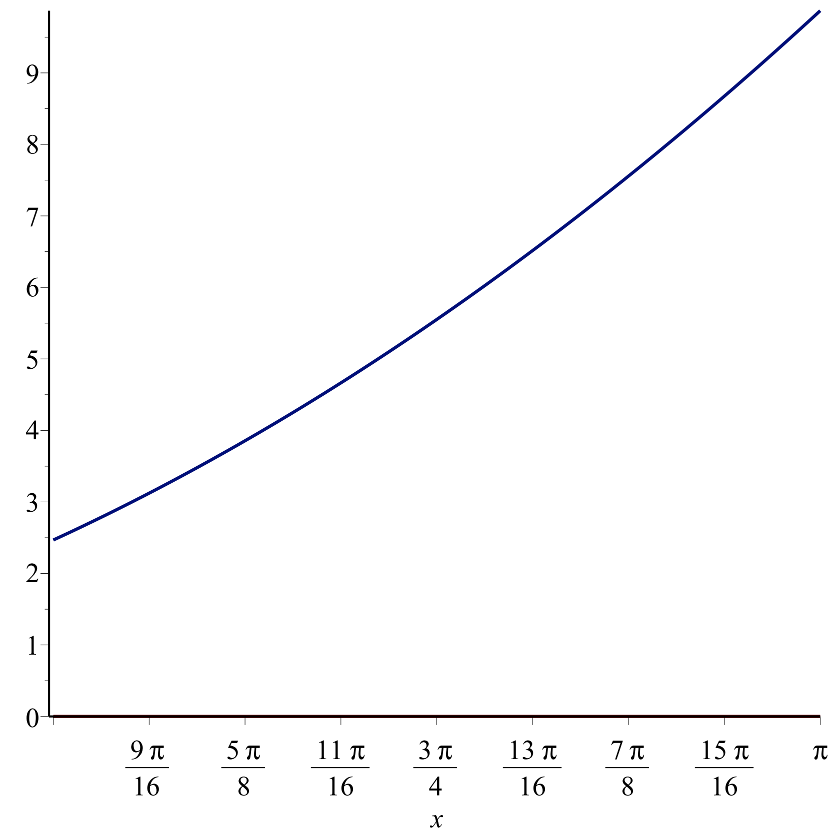
Like enkelt.

Eksempel 2

Finn $\iint \frac{1}{x} \cos\left(\frac{y}{x}\right) dA$ over R

R er området

$plot\left([0, x^2], x = \frac{\text{Pi}}{2} .. \text{Pi} \right);$



dydx:

$MultiInt\left(\frac{1}{x}\ \cdot\cos\left(\frac{y}{x}\right), y = 0\,..x^2, x = \frac{\mathrm{Pi}}{2}\,..\mathrm{Pi}, output = steps\right);$

$$\int_{\frac{\pi}{2}}^{\pi}\int_{0}^{x^2}\frac{\cos\left(\frac{y}{x}\right)}{x}\,\mathrm{d}y\,\mathrm{d}x$$

$$=\int_{\frac{\pi}{2}}^{\pi}\left(\sin\left(\frac{y}{x}\right)\Bigg|_{y=0\,..\,x^2}\right)\mathrm{d}x$$

$$=\int_{\frac{\pi}{2}}^{\pi}\sin(x)\,\mathrm{d}x$$

$$=-\cos(x)\Bigg|_{x=\frac{\pi}{2}\,..\,\pi}$$

$$1 \tag{3.1}$$

dxdy: Vi må dele opp R i to

$MultiInt\left(\frac{1}{x}\cos\left(\frac{y}{x}\right), x=\frac{\mathrm{Pi}}{2}\,..\mathrm{Pi}, y=0\,..\frac{\pi^2}{4}, output=steps\right);$

$$\int_{0}^{\frac{\pi^2}{4}} \int_{\frac{\pi}{2}}^{\pi} \frac{\cos\left(\frac{y}{x}\right)}{x} \, dx \, dy$$

$$= \int_{0}^{\frac{\pi^2}{4}} \left(-\mathrm{Ci}\left(\frac{y}{x}\right) \Bigg|_{x = \frac{\pi}{2} .. \pi} \right) dy$$

$$= \int_{0}^{\frac{\pi^2}{4}} \left(\mathrm{Ci}\left(\frac{2\,y}{\pi}\right) - \mathrm{Ci}\left(\frac{y}{\pi}\right) \right) dy$$

$$= \left(\frac{\pi \left(\frac{2\,\mathrm{Ci}\left(\frac{2\,y}{\pi}\right) y}{\pi} - \sin\left(\frac{2\,y}{\pi}\right) \right)}{2} - \pi \left(\frac{\mathrm{Ci}\left(\frac{y}{\pi}\right) y}{\pi} - \sin\left(\frac{y}{\pi}\right) \right) \right) \Bigg|_{y = 0 .. \frac{\pi^2}{4}}$$

$$\frac{1}{2} \sqrt{2}\, \pi + \frac{1}{4}\, \mathrm{Ci}\left(\frac{1}{2}\, \pi\right) \pi^2 - \frac{1}{4}\, \mathrm{Ci}\left(\frac{1}{4}\, \pi\right) \pi^2 - \frac{1}{2}\, \pi \qquad \mathbf{(3.2)}$$

$$\mathit{MultiInt}\left(\frac{1}{x} \cos\left(\frac{y}{x}\right), x = \mathrm{sqrt}(y) .. \mathrm{Pi}, y = \frac{\pi^2}{4} .. \pi^2, \mathit{output} = \mathit{steps} \right);$$

$$\int_{\frac{\pi^2}{4}}^{\pi^2}\int_{\sqrt{y}}^{\pi} \frac{\cos\left(\frac{y}{x}\right)}{x}\, dx\, dy$$

$$= \int_{\frac{\pi^2}{4}}^{\pi^2} \left(-\mathrm{Ci}\left(\frac{y}{x}\right)\Big|_{x=\sqrt{y}\,..\pi} \right) dy$$

$$= \int_{\frac{\pi^2}{4}}^{\pi^2}\int_{\sqrt{y}}^{\pi} \frac{\cos\left(\frac{y}{x}\right)}{x}\, dx\, dy$$

$$= \int_{\sqrt{y}}^{\pi} \sin\left(\frac{y}{x}\right) dx \Bigg|_{y=\frac{\pi^2}{4}\,..\pi^2}$$

$$-\frac{1}{2}\sqrt{2}\,\pi - \frac{1}{4}\,\mathrm{Ci}\left(\frac{1}{2}\pi\right)\pi^2 + \frac{1}{4}\,\mathrm{Ci}\left(\frac{1}{4}\pi\right)\pi^2 + \frac{1}{2}\pi + 1 \qquad \textbf{(3.3)}$$

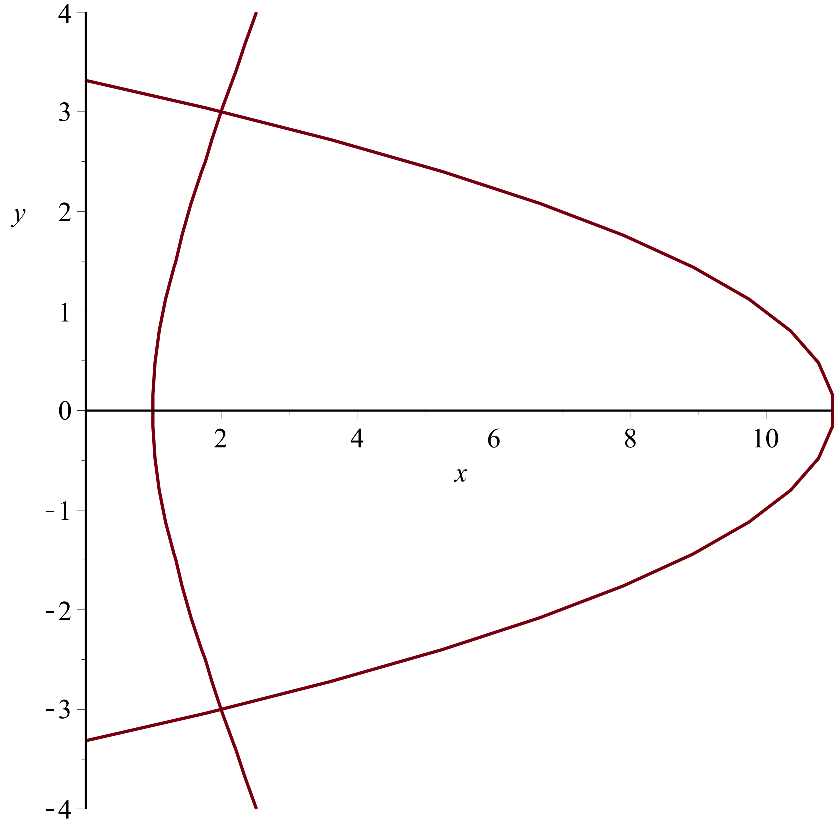
Hvis vi tar summen får vi 1.
dxdy var vanskeligere. Hvorfor?
- vi måtte dele opp R (ikke et stort problem, men tar mer tid)
- antideriverten var stygg! Det er et stort problem! dydx er en bedre valg nå

Eksempel 3 (2012 sommer / 3)

Finn $\iint x \, dA$ over R

R er området

implicitplot$\left([3 \cdot x^2 - y^2 = 3, x + y^2 = 11], x = 0\,..11, y = -4\,..4\right)$;



Skjæringspunktene er (2,3) og (2,-3)

$solve([3 \cdot x^2 - y^2 = 3, x + y^2 = 11])$

$$\{x=2, y=3\}, \{x=2, y=-3\}, \left\{x=-\frac{7}{3}, y=2\,RootOf(3\,\_Z^2-10)\right\} \tag{4.1}$$

dxdy:

$MultiInt\left(x, x = \mathrm{sqrt}\left(1+\frac{y^2}{3}\right)..11-y^2, y=-3..3, output=steps\right);$

$$\int_{-3}^{3}\int_{\frac{\sqrt{9+3y^2}}{3}}^{-y^2+11} x \, dx \, dy$$

$$= \int_{-3}^{3} \left(\left. \frac{x^2}{2} \right|_{x = \frac{\sqrt{9+3y^2}}{3} .. -y^2+11} \right) dy$$

$$= \int_{-3}^{3} \left(\frac{(-y^2+11)^2}{2} - \frac{1}{2} - \frac{y^2}{6} \right) dy$$

$$= \left. \left(60\,y + \frac{1}{10}\,y^5 - \frac{67}{18}\,y^3 \right) \right|_{y=-3..3}$$

$$\frac{1038}{5} \qquad \textbf{(4.2)}$$

dydx: (Vi må dele opp R i to)

$MultiInt(x, y = -\mathrm{sqrt}(3 \cdot x^2 - 3) .. \mathrm{sqrt}(3 \cdot x^2 - 3), x = 1..2, output = steps);$

$$\int_{1}^{2}\int_{-\sqrt{3x^2-3}}^{\sqrt{3x^2-3}} x \, dy \, dx$$

$$= \int_{1}^{2} \left(\left. y\,x \right|_{y = -\sqrt{3\,x^2-3} .. \sqrt{3\,x^2-3}} \right) dx$$

$$= \int_{1}^{2} 2\,x\sqrt{3\,x^2-3} \, dx$$

$$= \left. \frac{2\,(x-1)\,(x+1)\sqrt{3\,x^2-3}}{3} \right|_{x=1..2}$$

$$6 \qquad \textbf{(4.3)}$$

$MultiInt(x, y = -\mathrm{sqrt}(11 - x) .. \mathrm{sqrt}(11 - x), x = 2..11, output = steps);$

$$\int_{2}^{11}\int_{-\sqrt{11-x}}^{\sqrt{11-x}} x \, dy \, dx$$

$$= \int_{2}^{11} \left(y\,x \Big|_{y = -\sqrt{11-x} \,..\, \sqrt{11-x}} \right) dx$$

$$= \int_{2}^{11} 2\,x\,\sqrt{11-x}\, dx$$

$$= -\frac{4\,(22+3\,x)\,(11-x)^{3/2}}{15} \Bigg|_{x = 2\,..\,11}$$

$$\frac{1008}{5} \tag{4.4}$$

Hvis vi tar summen får vi 1038/5.
dydx er en litt vanskeligere. Hvorfor?
- vi måtte dele opp R (ikke et stort problem, men tar mer tid)
- antideriverten var ikke så enkelt men det er ok. Prøv å finne med hånd.
    - i det første integralet (i dydx) merker vi at 2x = (x^2)' derfor vi kan finne et uttrykk
    - i det andre integralet (i dydx) bruker vi partiell integrasjon faktisk.

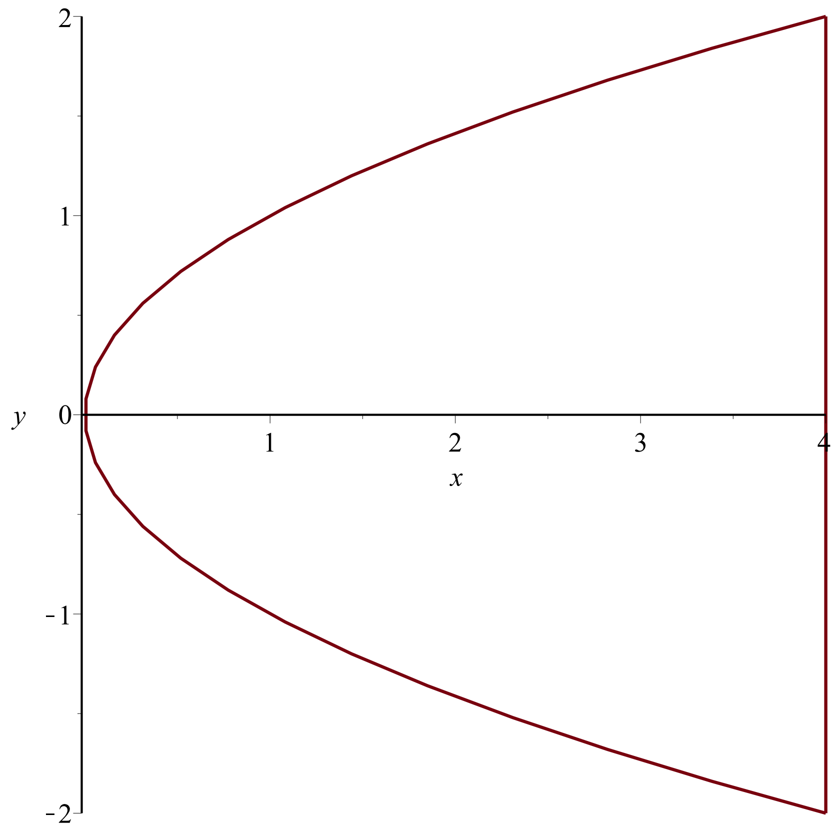
Siden begge problemer (grensene, antiderivertene) tar mer tid ved dydx, bruker vi dxdy

Eksempel 4

Finn arelaet til R, dvs $\iint 1 \, dA$ over R

R er området

implicitplot$\left([x = 4, x = y^2], x = 0\,..4, y = -2\,..2 \right)$;



dxdy:

MultiInt$(1, x = y^2..4, y = -2..2, output = steps)$;

$$\int_{-2}^{2}\int_{y^2}^{4} 1 \, dx \, dy$$

$$= \int_{-2}^{2} \left(x \Big|_{x = y^2..4} \right) dy$$

$$= \int_{-2}^{2} (4 - y^2) \, dy$$

$$= \left(4y - \frac{1}{3} y^3 \right) \Big|_{y = -2..2}$$

$$\frac{32}{3} \qquad \textbf{(5.1)}$$

dydx:

```
MultiInt(1, y=-sqrt(x) ..sqrt(x), x=0 ..4, output=steps);
```

$$\int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} 1 \, dy \, dx$$

$$= \int_0^4 \left(y \Big|_{y = -\sqrt{x} \,..\, \sqrt{x}} \right) dx$$

$$= \int_0^4 2\sqrt{x} \, dx$$

$$= \left. \frac{4 x^{3/2}}{3} \right|_{x = 0 \,..\, 4}$$

$$\frac{32}{3} \tag{5.2}$$

det er like enkelt på en måte. Uansett, legg merke til at oppgaven gir oss grensene til dxdy med en gang. Til å finne grensene til dydx må vi jobbe en litt.
Derfor tar dxdy mindre tid.