# Start

Vi starter på nytt

***restart;***

Vi lader inn kommandopakken plots

***with(plots);***

[*animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot*] **(1.1)**

Vi lader inn kommandopakken VectorCalculus

***with(Student[VectorCalculus]);***

[*&x*, \`*\`, \`+\`, \`-\`, \`.\`, <,>, <|>, *About, ArcLength, BasisFormat, Binormal, ConvertVector, CrossProduct, Curl, Curvature*, D, *Del, DirectionalDiff, Divergence, DotProduct, FlowLine, Flux, GetCoordinates, GetPVDescription, GetRootPoint, GetSpace, Gradient, Hessian, IsPositionVector, IsRootedVector, IsVectorField, Jacobian, Laplacian, LineInt, MapToBasis, Nabla, Norm, Normalize, PathInt, PlotPositionVector, PlotVector, PositionVector, PrincipalNormal, RadiusOfCurvature, RootedVector, ScalarPotential, SetCoordinates, SpaceCurve, SpaceCurveTutor, SurfaceInt, TNBFrame, Tangent, TangentLine, TangentPlane, TangentVector, Torsion, Vector, VectorField, VectorFieldTutor, VectorPotential, VectorSpace, diff, evalVF, int, limit, series*] **(1.2)**

Vi lader inn kommandopakken MultivariateCalculus

***with(Student[MultivariateCalculus]);***

[*ApproximateInt, ApproximateIntTutor, CenterOfMass, ChangeOfVariables, CrossSection, CrossSectionTutor, Del, DirectionalDerivative, DirectionalDerivativeTutor, FunctionAverage, Gradient, GradientTutor, Jacobian, LagrangeMultipliers, MultiInt, Nabla, Revert, SecondDerivativeTest, SurfaceArea, TaylorApproximation, TaylorApproximationTutor*] **(1.3)**

# Eksamensoppgave 2006v / 3

Vi jobber med f (finn maksima/minima for f)

$$f := (x, y) \rightarrow \left(\frac{x-1}{2}\right)^2 + y^2;$$

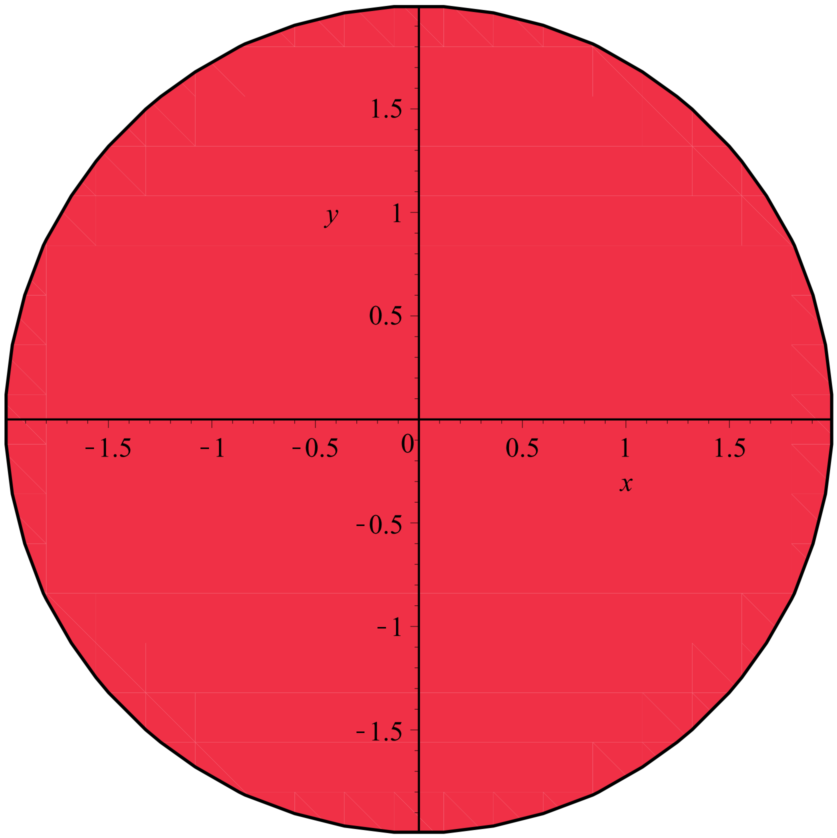
$$(x, y) \rightarrow \left(1\ \frac{1}{2}\ x + \mathit{Student{:}\text{-}VectorCalculus{:}\text{-}}\text{\`-\`}\left(1\ \frac{1}{2}\right)\right)^2 + y^2 \qquad \textbf{(2.1)}$$

Definisjonsmengden

$$DEFOMRADE := implicitplot\left(x^2 + y^2 \leq 4, x = -3..3, y = -3..3, filledregions = true\right);$$

$$PLOT(\ldots) \tag{2.2}$$

***display(DEFOMRADE)***




Gradienten til f

***Gradient(f(x, y));***

$$\left(\frac{1}{2}x - \frac{1}{2}\right)\bar{e}_x + 2\,y\bar{e}_y \tag{2.3}$$

randen til D // bibetingelse (finn maksima/minima for f under bibetingelsen g= 0)

$g := (x, y) \rightarrow x^2 + y^2 - 4;$

$$(x, y) \rightarrow x^2 + y^2 + (-4) \tag{2.4}$$

Vi lager forskjellige bilddeler:

grafen til f

***GRAFENF := plot3d(f(x, y), x = -3 ..3, y = -3 ..3) :***

grafen til g

***GRAFENG := implicitplot3d(g(x, y) = 0, x = -2 ..2, y = -2 ..2, z = -4 ..10, style = surface, color = cyan, transparency = 0.5) :***

snittkurve mellom flaten og bibetingelsen

***SNITTKURVEa := SpaceCurve(⟨t, sqrt(4 − t^2), f(t, sqrt(4 − t^2))⟩, t = -2 ..2, color = red, thickness = 2) :***

$SNITTKURVEb := SpaceCurve(\langle t, -\mathrm{sqrt}(4-t^2), f(t, -\mathrm{sqrt}(4-t^2))\rangle, t = -2..2, color = red, thickness = 2)$ :

Nivåkurvene til f tilsv f = 0, f =1/4, f = 9/4, f = 13/3

$NIVAKURVERFa := contourplot\left(f(x, y), x = -3..3, y = -3..3, filledregions = true, coloring = [\text{"White"}, \text{"DarkViolet"}], contours = \left[0, \frac{1}{4}, \frac{9}{4}, \frac{13}{3}\right]\right)$ :

Nivåkurvene til f tilsv 20 ulike verdier

$NIVAKURVERFb := contourplot(f(x, y), x = -3..3, y = -3..3, filledregions = true, coloring = [\text{"White"}, \text{"DarkViolet"}], contours = 20, transparency = 0.5)$ :

Bibetingelsen g = 0

$BIBETINGELSEG := implicitplot(g(x, y) = 0, x = -3..3, y = -3..3)$ :

Gradientfeltet til f

$GRADPLOTF := gradplot(f(x, y), x = -3..3, y = -3..3, color = \text{'blue'})$ :

En vektor med retningen av GradF, størrelse 0.5

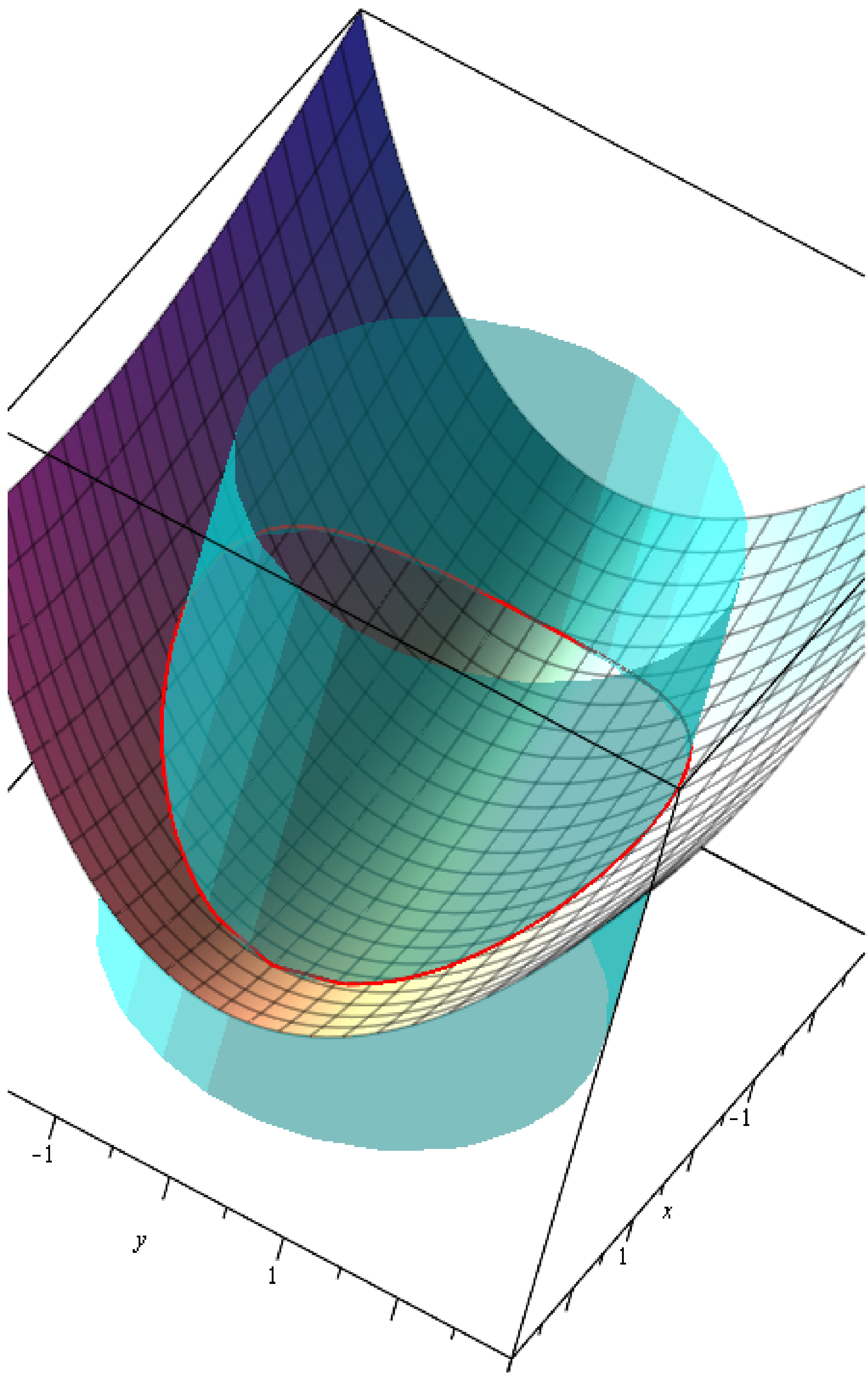
$GRADIENTF := (u, v) \rightarrow arrow(\langle u, v\rangle, subs(x = u, y = v, Gradient(f(x, y))), length = 0.5, shape = double\_arrow)$ :

En vektor med retningen av GradG, størrelse 0.3

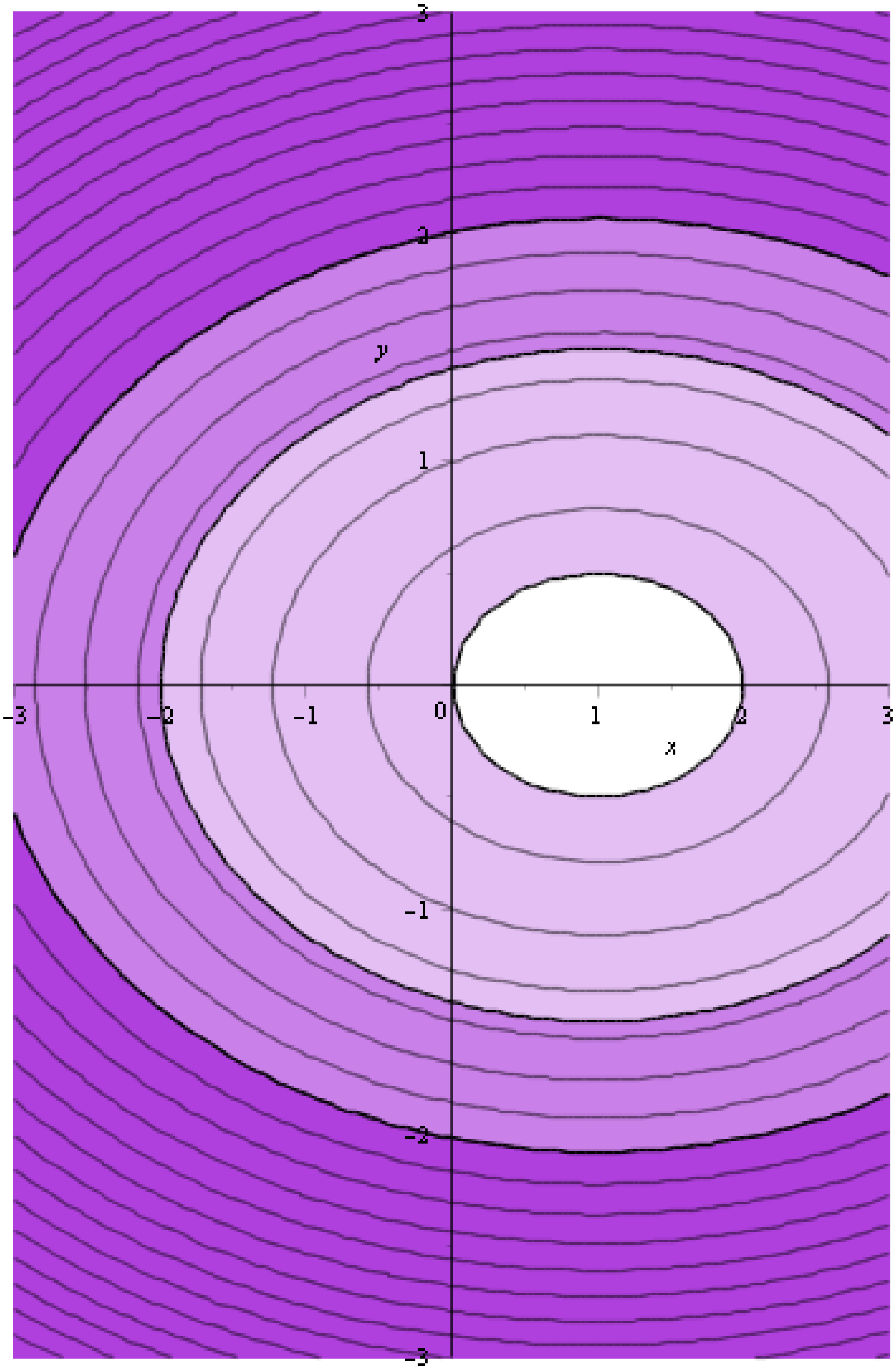
$GRADIENTG := (u, v) \rightarrow arrow(\langle u, v\rangle, subs(x = u, y = v, Gradient(g(x, y))), length = 0.3, shape = arrow, color = red)$ :

Grafen til f sammen med bibetingelsen g = 0 og snittkurven

$display(GRAFENF, GRAFENG, SNITTKURVEa, SNITTKURVEb, axes = boxed)$

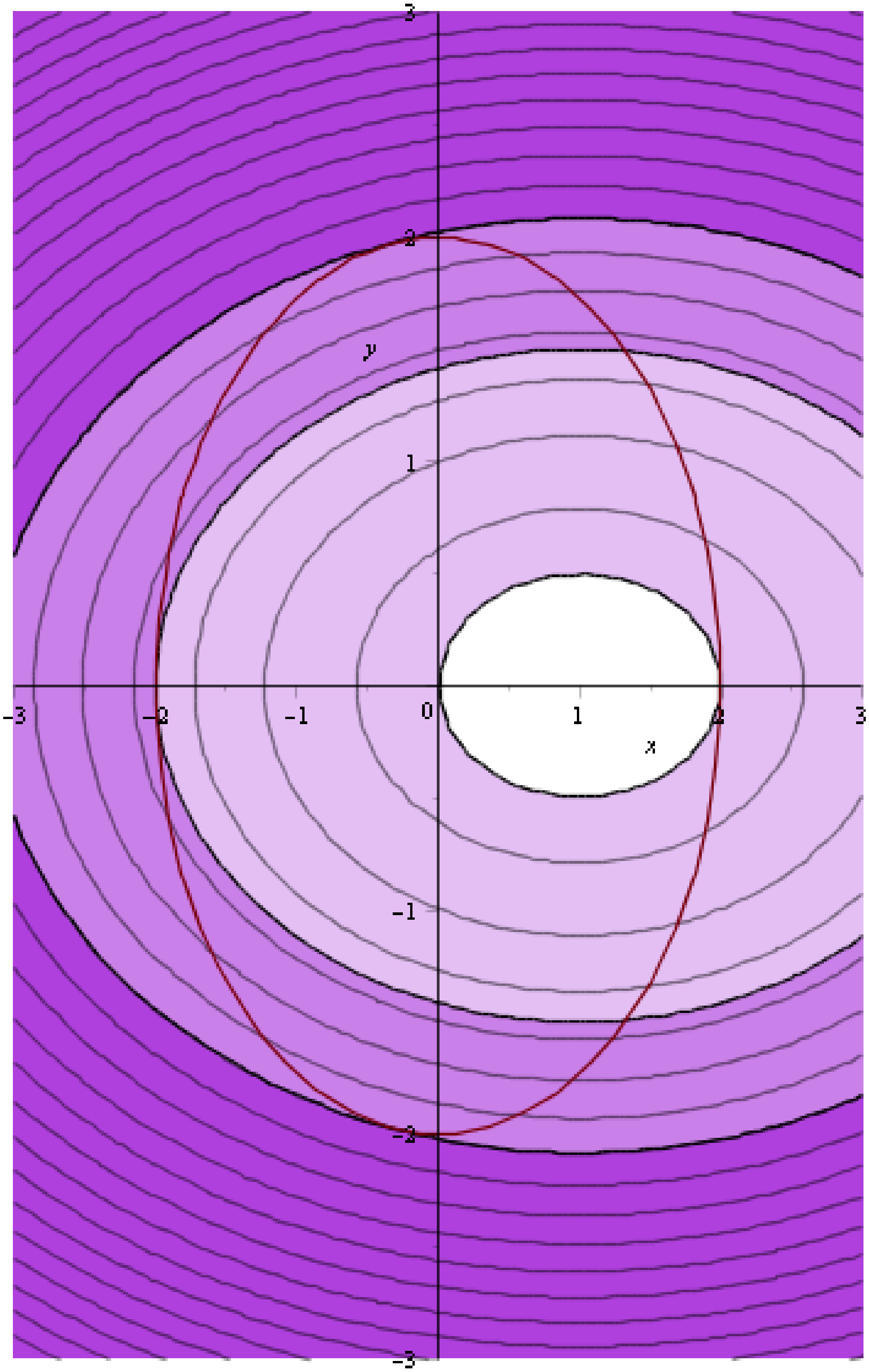

-1
y
1
-1
x
1

Nivåkurvene til f på xy-planet (ellipser)
***display***(***NIVAKURVERFa***, ***NIVAKURVERFb***)



3
2
y
1
–1
–2
–3
–3
–2
–1
0
1
2
3
x

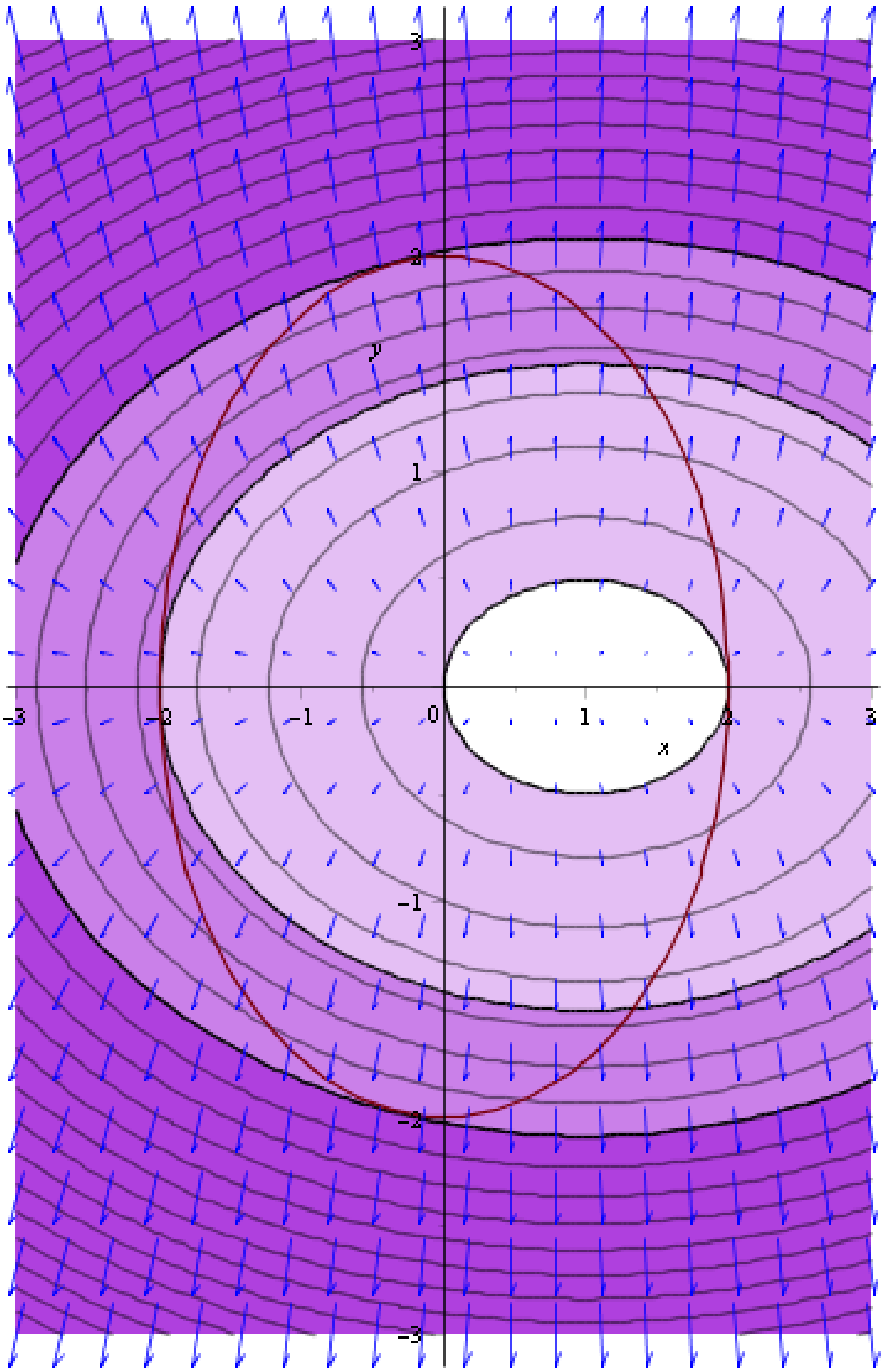
Nivåkurvene til f og bibetingelsen på xy-planet
$display(NIVAKURVERFa, NIVAKURVERFb, BIBETINGELSEG)$



3
2
y
1
0
-1
-2
-3
-3
-2
-1
1
2
3
x

Nivåkurvene til f, bibetingelsen på xy-planet, gradientfeltet til f. GradF står vinkelrett på nivåkurvene til f

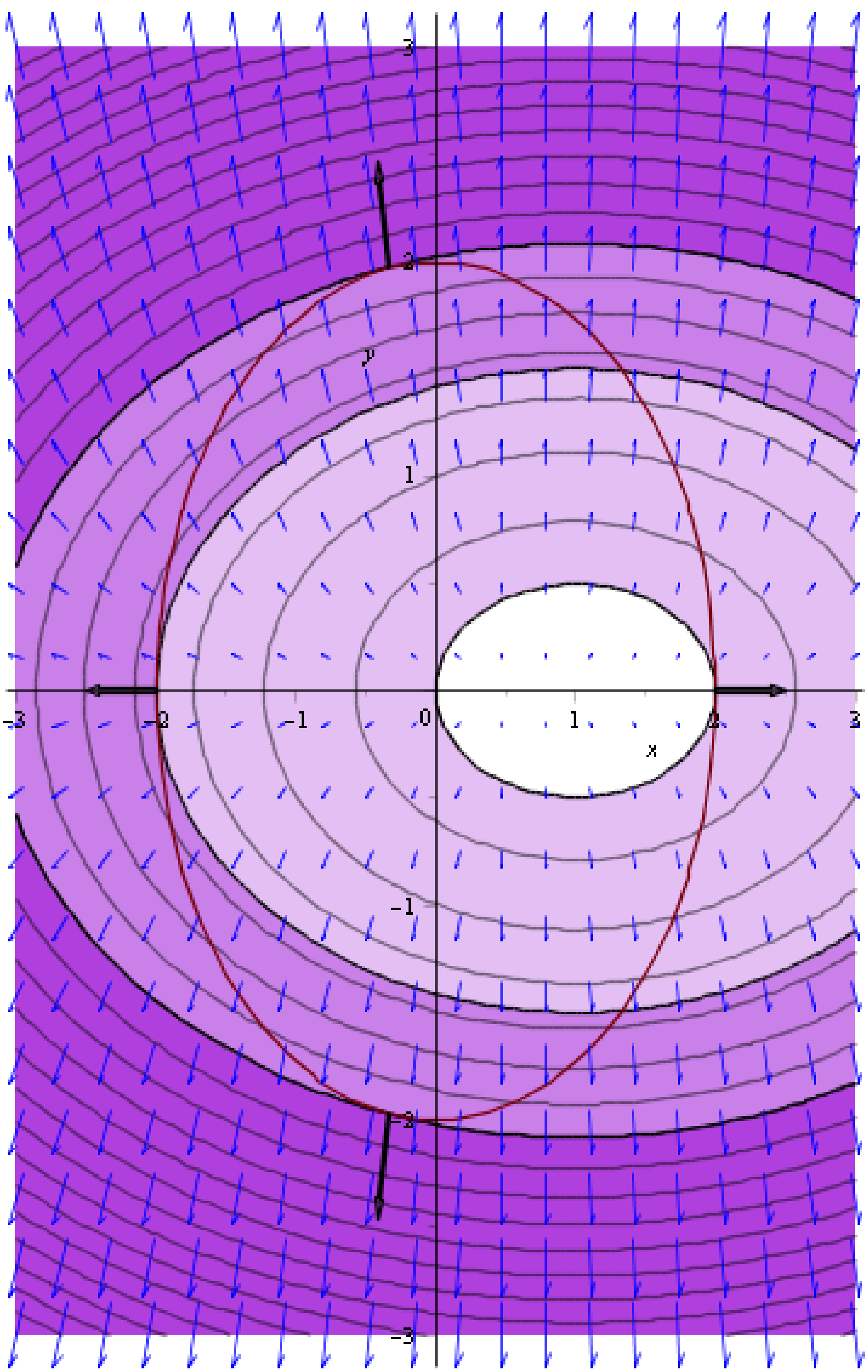
*display*(*NIVAKURVERFa*, *NIVAKURVERFb*, *BIBETINGELSEG*, *GRADPLOTF*)



x
y
-3
-2
-1
0
1
2
3

GradF står vinkelrett på kurven g = 0 i 4 punkter

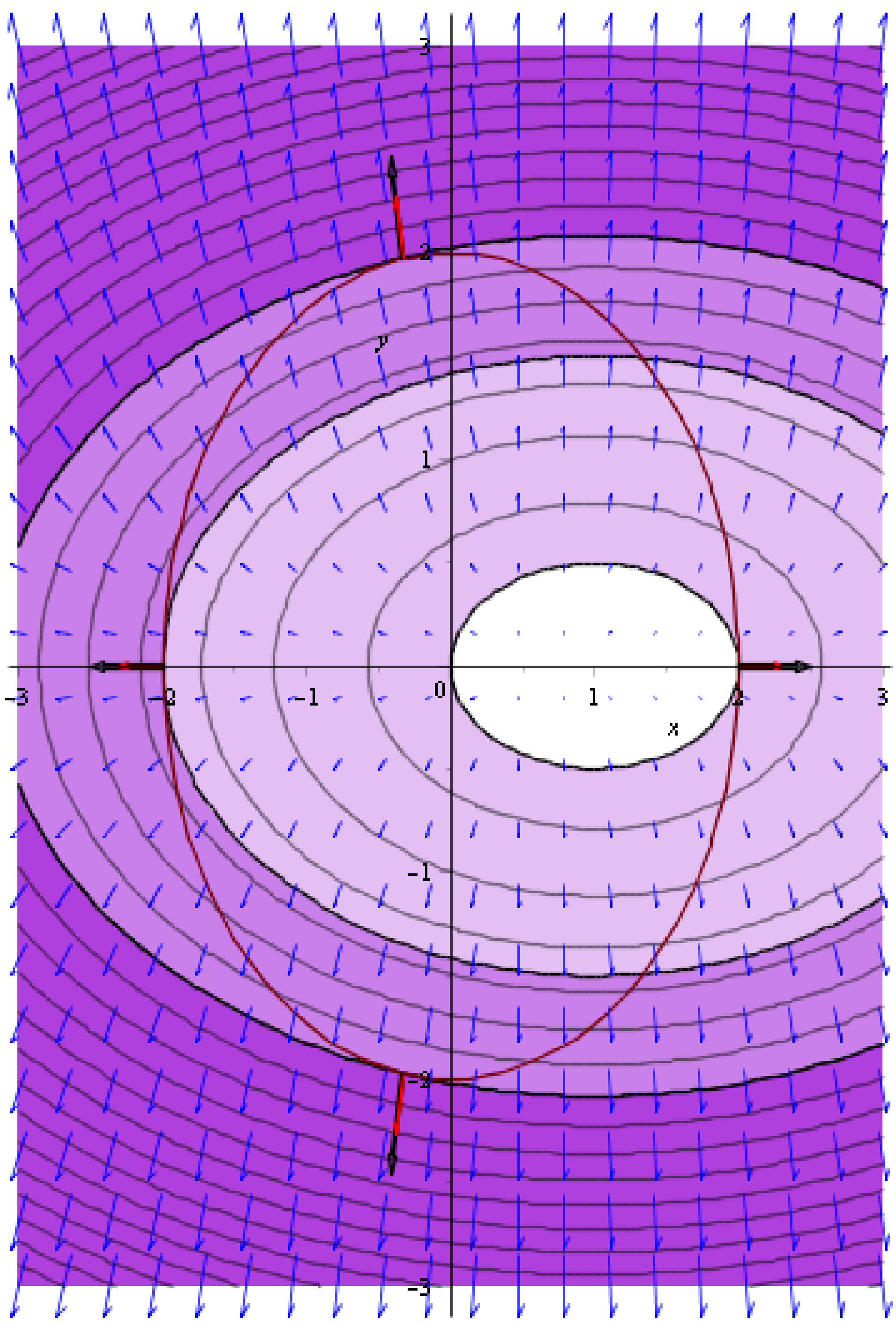
$$\mathit{display}\left(\mathit{NIVAKURVERFa}, \mathit{NIVAKURVERFb}, \mathit{BIBETINGELSEG}, \mathit{GRADPLOTF}, \mathit{GRADIENTF}(-2, 0), \mathit{GRADIENTF}(2, 0), \mathit{GRADIENTF}\left(-\frac{1}{3}, \frac{\mathrm{sqrt}(35)}{3}\right), \mathit{GRADIENTF}\left(-\frac{1}{3}, -\frac{\mathrm{sqrt}(35)}{3}\right)\right)$$



x
y

Siden g = 0 er en nivåkurve til g(x,y), derfor står også GradG vinkelrett på kurven g = 0. Derfor er GradF parallelt med GradG.

$$\mathit{display}\left(\mathit{NIVAKURVERFa}, \mathit{NIVAKURVERFb}, \mathit{BIBETINGELSEG}, \mathit{GRADPLOTF}, \mathit{GRADIENTF}(-2, 0), \mathit{GRADIENTF}(2, 0), \mathit{GRADIENTF}\left(-\frac{1}{3}, \frac{\mathrm{sqrt}(35)}{3}\right), \mathit{GRADIENTF}\left(-\frac{1}{3}, -\frac{\mathrm{sqrt}(35)}{3}\right), \mathit{GRADIENTG}(-2, 0), \mathit{GRADIENTG}(2, 0), \mathit{GRADIENTG}\left(-\frac{1}{3}, \frac{\mathrm{sqrt}(35)}{3}\right), \mathit{GRADIENTG}\left(-\frac{1}{3}, -\frac{\mathrm{sqrt}(35)}{3}\right)\right)$$



3
2
y
1
-1
-2
-3
-3
-2
-1
0
1
2
3
x

# Cobb-Douglas

Vi jobber med f (finn maksima/minima for f, definisjonsmengden: x>0,y>0)

$f := (x, y) \rightarrow 400 \cdot x^{0.25} \cdot y^{0.5};$

$$(x, y) \rightarrow 400\, x^{0.25}\, y^{0.5} \quad (3.1)$$

Bibetingelsen g = 0 (finn maksima/minima for f under bibetingelsen g = 0)

$g := (x, y) \rightarrow 100 \cdot x + 50 \cdot y - 150;$

$$(x, y) \rightarrow 100\, x + 50\, y + (-150) \quad (3.2)$$

Gradientene

$Gradient(f(x, y));$

$$\frac{100.00\, y^{0.5}}{x^{0.75}}\, \bar{e}_x + \frac{200.0\, x^{0.25}}{y^{0.5}}\, \bar{e}_y \quad (3.3)$$

$Gradient(g(x, y));$

$$100\bar{e}_x + 50\bar{e}_y \quad (3.4)$$

Lagranges multiplikatormetode == Finn de kritiske punktene til F

$F := (x, y, \lambda) \rightarrow f(x, y) - \lambda \cdot g(x, y);$

$$(x, y, \lambda) \rightarrow f(x, y) + Student\text{:-}VectorCalculus\text{:-}\grave{}\text{-}\grave{}(\lambda\, g(x, y)) \quad (3.5)$$

Vi kan gjør det med Maple: (ikke så enkelt å se på, http://www.maplesoft.com/support/help/Maple/view.aspx?path=Task/SecondDerivativeTest)
Løsningen er x = 0.5, y = 2

$VAR := [x, y, \lambda]:$

$temp := remove(has, solve(convert(Gradient(F(x, y, \lambda)), list), VAR, Explicit), I):$

$convert(\{seq(eval(VAR, temp[k]), k = 1 ..nops(temp))\}, list);$

$$[[0.5000000000, 2., 2.378414230]] \quad (3.6)$$

Eller en enda enklere metod:

$LagrangeMultipliers(f(x, y), [g(x, y)], [x, y]);$

$$[0.5000000000, 2.] \quad (3.7)$$

Funksjonsverdien her er

$f(0.5, 2)$

$$475.6828459 \quad (3.8)$$

Det er en selvfølge at det må være en maksima, siden når vi går måt randen til definisjonsområdet (x -> 0 eller y -> 0, da f -> 0)

Noe bilder som skal brukes
Grafen til f

$GRAFENF := plot3d(f(x, y), x = 0 ..5, y = 0 ..5):$

Bibetingelsen g = 0

$GRAFENG := implicitplot3d(g(x, y) = 0, x = 0 ..5, y = 0 ..5, z = 0 ..1200, style = surface, color = cyan, transparency = 0.5):$

Snittkurven

$SNITTKURVE := SpaceCurve\left(\left\langle t, \frac{150 - 100 \cdot t}{50}, f\left(t, \frac{150 - 100 \cdot t}{50}\right)\right\rangle, t = 0 ..1.5, color = red, thickness = 2\right):$

Noe nivåkurver til f

```
NIVAKURVERFa := contourplot(f(x, y), x = 0 ..5, y = 0 ..5, filledregions = true, coloring
    = ["White", "DarkViolet"], contours = [0, 110, 220, 330, 475.6]) :
NIVAKURVERFb := contourplot(f(x, y), x = 0 ..5, y = 0 ..5, filledregions = true, coloring
    = ["White", "DarkViolet"], contours = 20, transparency = 0.5) :
```

Bibetingelsen g = 0

```
BIBETINGELSEG := implicitplot(g(x, y) = 0, x = 0 ..5, y = 0 ..5) :
```

Gradientfeltet til f

```
GRADPLOTF := gradplot(f(x, y), x = 0 ..5, y = 0 ..5, color ='blue') :
```

En vektor med størrelse 0.5, retningen av GradF

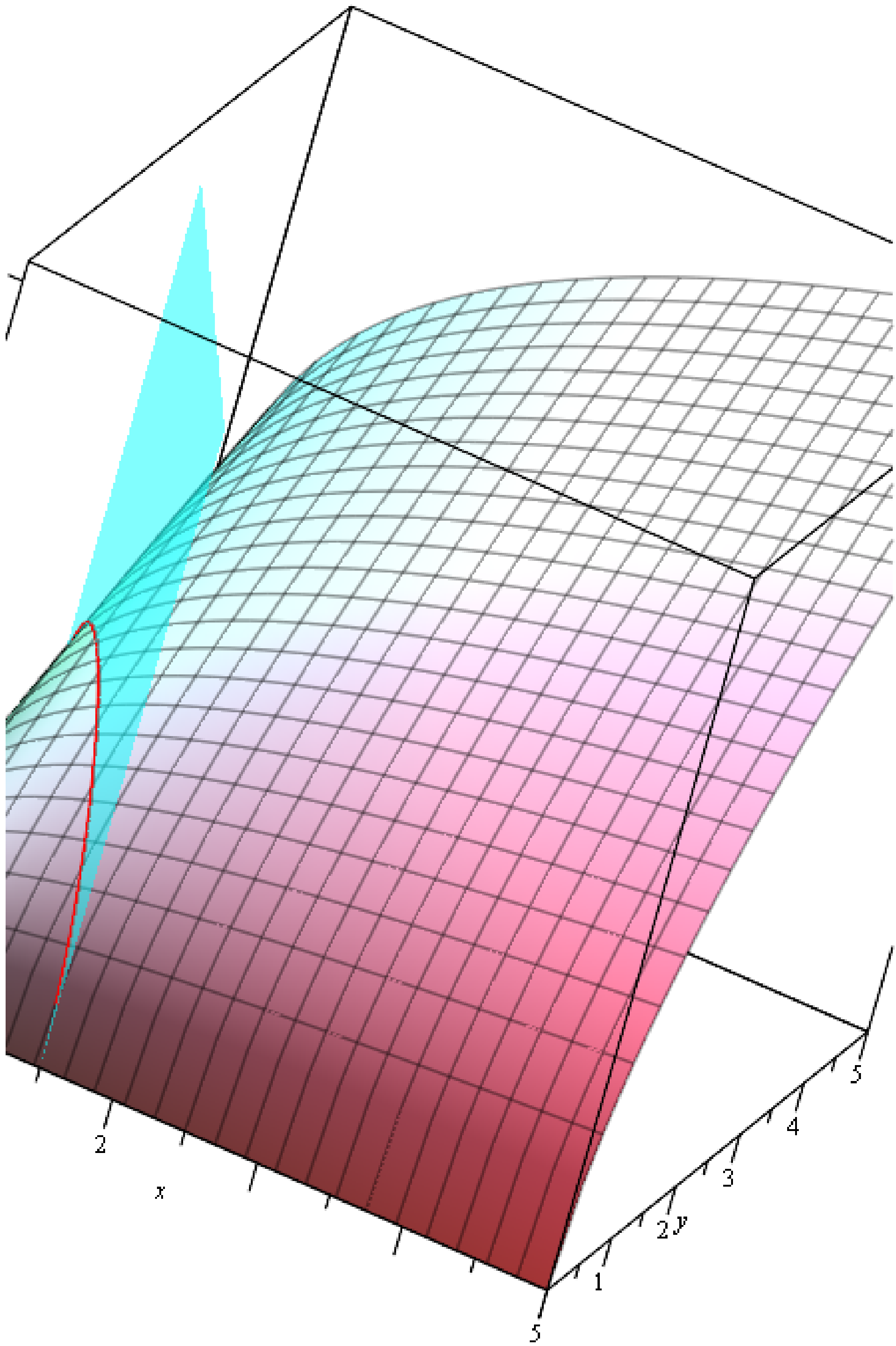
```
GRADIENTF := (u, v) → arrow(⟨u, v⟩, subs(x = u, y = v, Gradient(f(x, y))), length = 0.5, shape
    = double_arrow) :
```

En vektor med størrelse 0.3, retningen av GradG

```
GRADIENTG := (u, v) → arrow(⟨u, v⟩, subs(x = u, y = v, Gradient(g(x, y))), length = 0.3, shape
    = arrow, color = red) :
```

Graden til g, bibetingelsen, snittkurven

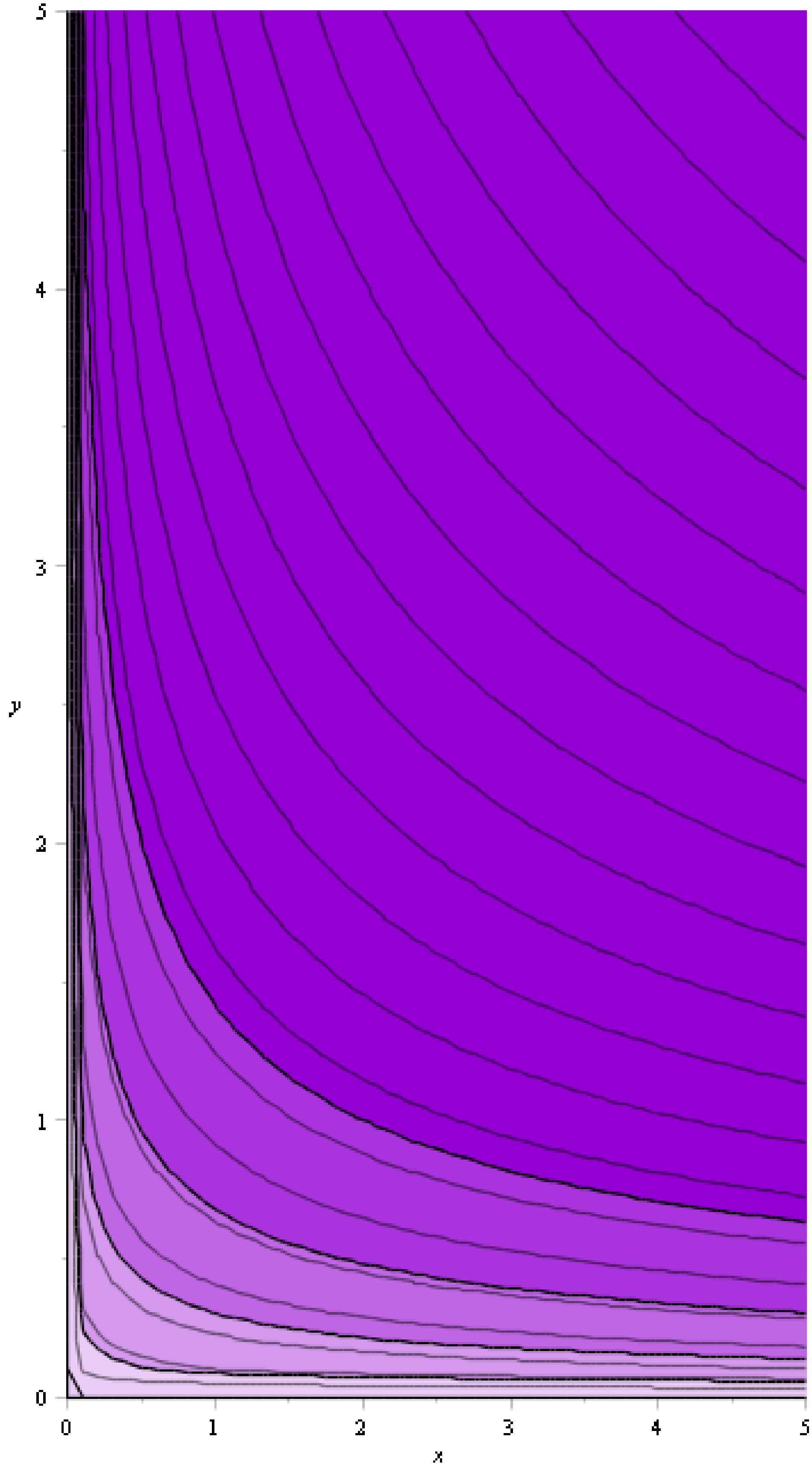
```
display(GRAFENF, GRAFENG, SNITTKURVE, axes = boxed)
```



2
x
5
1
2
y
3
4
5

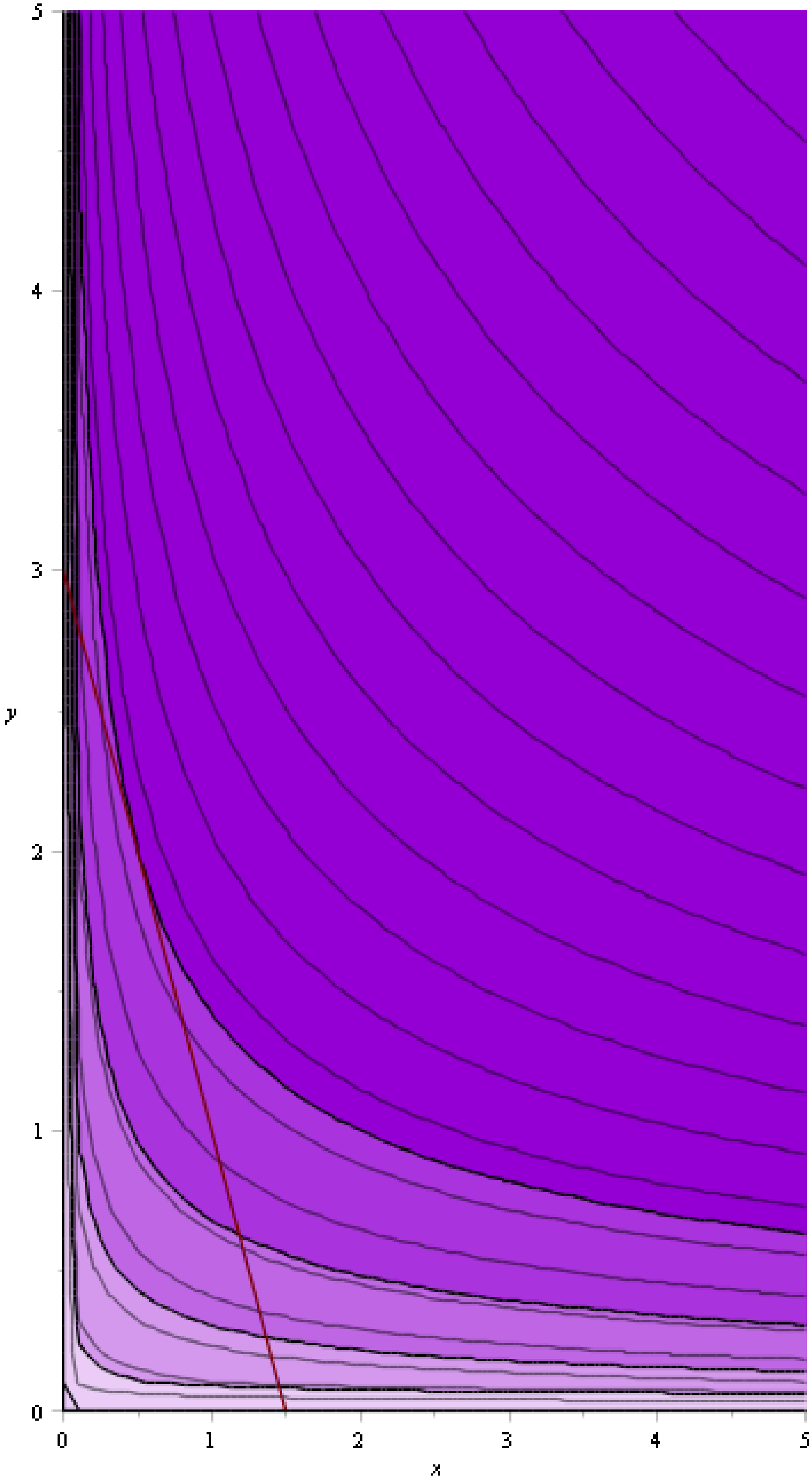
Nivåkurvene til f

***display***($\boldsymbol{NIVAKURVERFa}$, $\boldsymbol{NIVAKURVERFb}$)



5
4
3
2
1
0
y
0
1
2
3
4
5
x

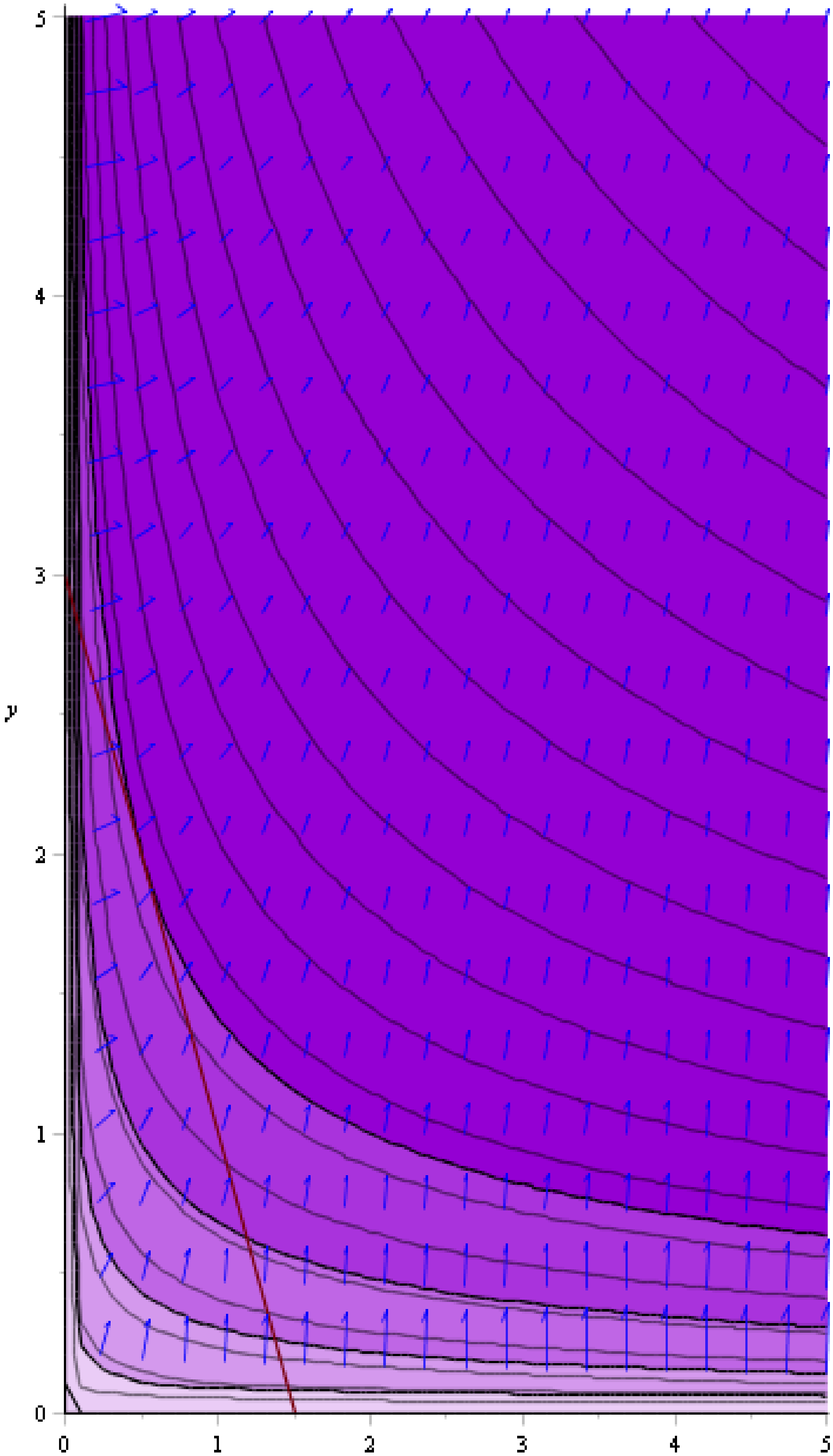
Nivåkurvene til f og bibetingelsen
***display***(***NIVAKURVERFa***, ***NIVAKURVERFb***, ***BIBETINGELSEG***)



5
4
3
2
1
0
y
0
1
2
3
4
5
x

Nivåkurvene, bibetingelsen og gradientfeltet til f

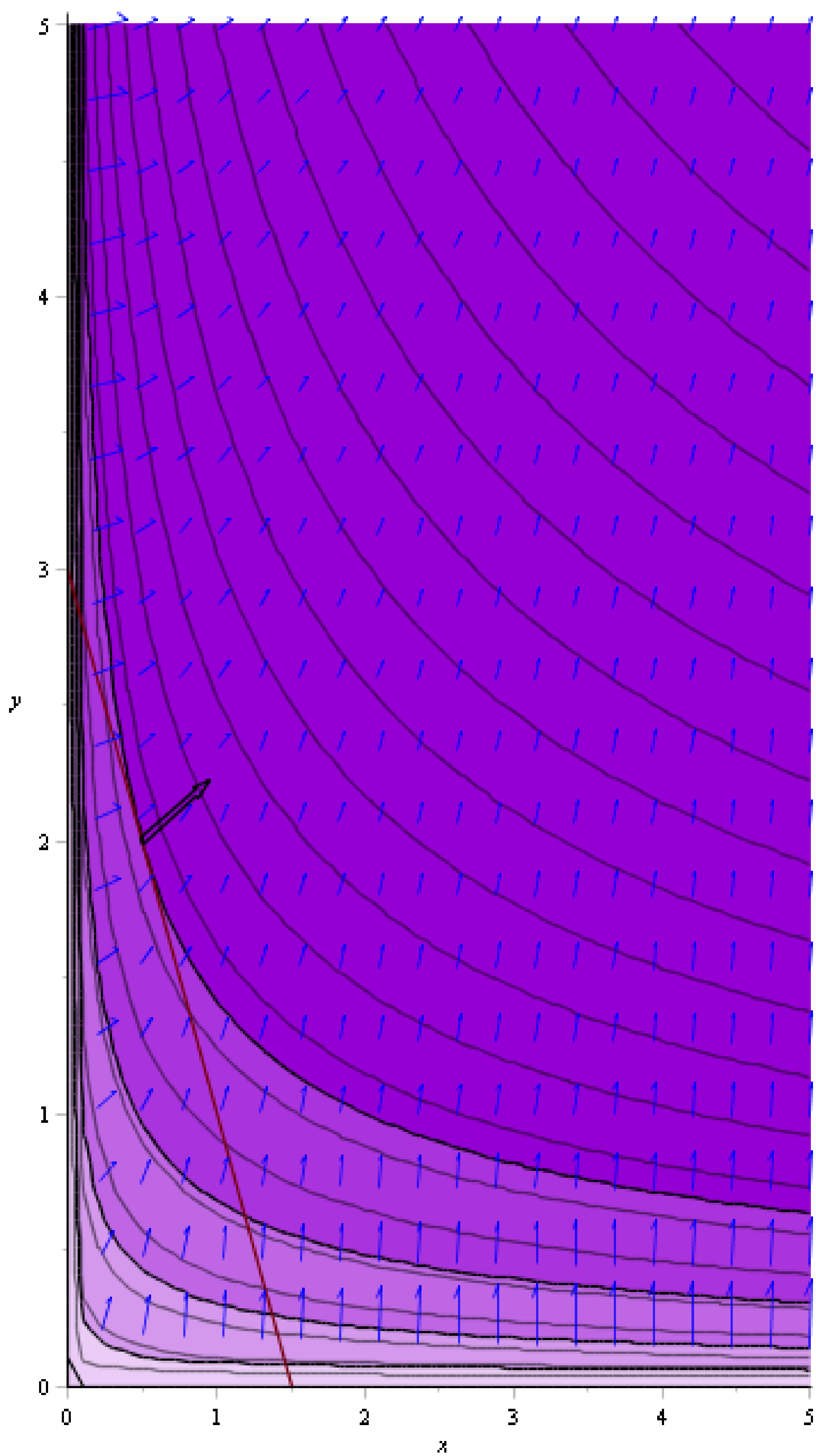
***display***(***NIVAKURVERFa***, ***NIVAKURVERFb***, ***BIBETINGELSEG***, ***GRADPLOTF***)



5
4
3
y
2
1
0
0
1
2
3
4
5
x

Nivåkurvene, bibetingelsen, gradientfeltet, gradienten til f i (0.5,2)

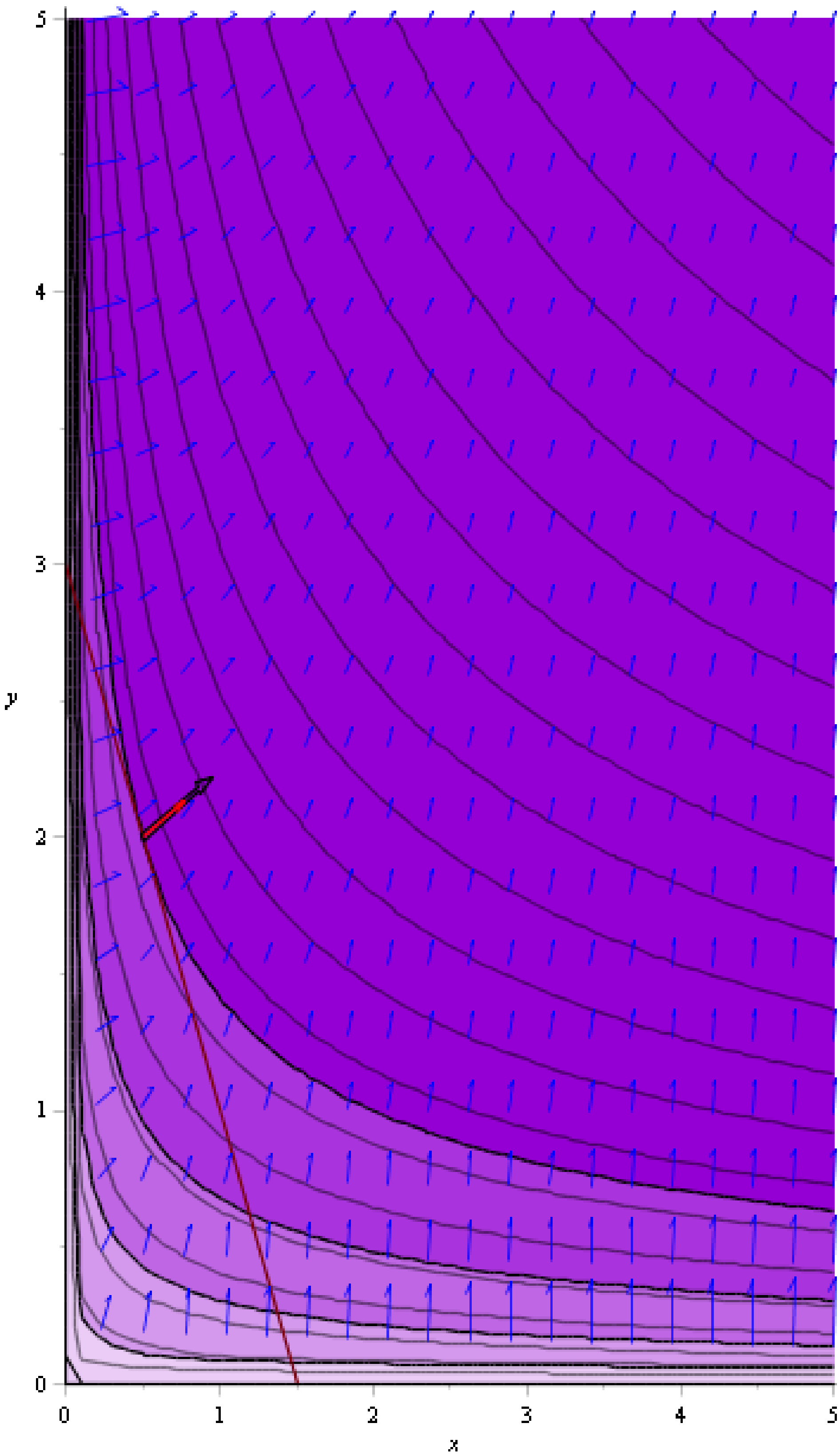
$$display\left(NIVAKURVERFa, NIVAKURVERFb, BIBETINGELSEG, GRADPLOTF, GRADIENTF\left(\frac{1}{2}, 2\right)\right)$$



5
4
3
y
2
1
0
0
1
2
3
4
5
x

Nivåkurvene, bibetingelsen, gradientfeltet, GradF, GradG

$$\boldsymbol{display\left(NIVAKURVERFa, NIVAKURVERFb, BIBETINGELSEG, GRADPLOTF, GRADIENTF\left(\frac{1}{2}, 2\right), GRADIENTG\left(\frac{1}{2}, 2\right)\right)}$$



5
4
3
2
1
0
y
0
1
2
3
4
5
x

# To bibetingelser

Finn punktene som ligger på skjæringskurven mellom x+y+z=12 og z=x^2+y^2 og er nærmest eller lengst til origo

Vi jobber med f (finn minima/maksima for f, dvs min/max til avstanden fra origo)

$f := (x, y, z) \to x^2 + y^2 + z^2;$

$$(x, y, z) \to x^2 + y^2 + z^2 \qquad \textbf{(4.1)}$$

Under to bibetingelser g1 = 0, g2 = 0

g1 = 0 gir oss et plan

$g1 := (x, y, z) \to x + y + z - 12;$

$$(x, y, z) \to x + y + z + (-12) \qquad \textbf{(4.2)}$$

g2 = 0 gir oss et parabel

$g2 := (x, y, z) \to x^2 + y^2 - z;$

$$(x, y, z) \to x^2 + y^2 + \mathit{Student{:}\text{-}VectorCalculus}{:}\text{-}\text{`-`}(z) \qquad \textbf{(4.3)}$$

De skjærer hverendre i en ellipse

Vi bruker Lagranges multiplikatormetode == vi finner de kritiske punktene til F

$F := (x, y, z, \lambda, \mu) \to f(x, y, z) - \lambda \cdot g1(x, y, z) - \mu \cdot g2(x, y, z);$

$$(x, y, z, \lambda, \mu) \to f(x, y, z) + \mathit{Student{:}\text{-}VectorCalculus}{:}\text{-}\text{`-`}(\lambda\, g1(x, y, z)) + \mathit{Student{:}\text{-}VectorCalculus}{:}\text{-}\text{`-`}(\mu\, g2(x, y, z)) \qquad \textbf{(4.4)}$$

Maple løser den slik:

$VAR := [x, y, z, \lambda, \mu]:$

$temp := remove(has, solve(convert(Gradient(F(x, y, z, \lambda, \mu)), list), VAR, Explicit), I):$

$convert(\{seq(eval(VAR, temp[k]), k = 1..nops(temp))\}, list);$

$$\left[\left[-3, -3, 18, \frac{222}{5}, \frac{42}{5}\right], \left[2, 2, 8, \frac{68}{5}, -\frac{12}{5}\right]\right] \qquad \textbf{(4.5)}$$

Eller en enda enklere metod. Vi får 2 komplekse og 2 reelle løsninger, siden vi jobber i rommet vi trenger bare de reelle løsningene.

$LagrangeMultipliers(f(x, y, z), [g1(x, y, z), g2(x, y, z)], [x, y, z]);$

$$\left[\frac{25}{4} - \frac{1}{4} \cdot I\sqrt{629}, \frac{25}{4} + \frac{1}{4} \cdot I\sqrt{629}, -\frac{1}{2}\right], \left[\frac{25}{4} + \frac{1}{4} \cdot I\sqrt{629}, \frac{25}{4} - \frac{1}{4} \cdot I\sqrt{629}, -\frac{1}{2}\right], [-3, -3, 18], [2, 2, 8] \qquad \textbf{(4.6)}$$

De kritiske punktene er (-3, -3, 18) og (2, 2, 8)

$f(-3, -3, 18);$

$f(2, 2, 8);$

$$342$$

$$72 \qquad \textbf{(4.7)}$$

Vi kan se at (-3, -3, 18) har det største avstand fra origo og (2, 2, 8) har det minste på ellipsen.

Bildet:
Bibetingelsen g1 = 0 (et plan)

$BIBETINGELSE1 := implicitplot3d(g1(x, y, z) = 0, x = -5..5, y = -5..5, z = -5..20, color = cyan, transparency = 0.5, style = surface) :$

Bibetingelsen g2 = 0 (en paraboloide)

$BIBETINGELSE2 := implicitplot3d(g2(x, y, z) = 0, x = -5..5, y = -5..5, z = -5..20, color = blue, grid = [25, 25, 25], style = surface) :$

En vektor med retningen av GradG1 og størrelse 1

$GRADIENTG1 := (u, v, w) \rightarrow arrow(\langle u, v, w\rangle, subs(x = u, y = v, z = w, Gradient(g1(x, y, z))), length = 1, shape = arrow, color = red) :$

En vektor med retningen av GradG2 og størrelse 1

$GRADIENTG2 := (u, v, w) \rightarrow arrow(\langle u, v, w\rangle, subs(x = u, y = v, z = w, Gradient(g2(x, y, z))), length = 1, shape = arrow, color = magenta) :$

En vektor med retningen av GradF og størrelse 1

$GRADIENTF := (u, v, w) \rightarrow arrow(\langle u, v, w\rangle, subs(x = u, y = v, z = w, Gradient(f(x, y, z))), length = 1, shape = arrow, color = green) :$

g1 = 0, g2 = 0, Gradientene i (2,2,8). Merk at GradF er på planet spennet av GradG1, GradG2 i dette punktet

$display(BIBETINGELSE2, BIBETINGELSE1, GRADIENTG1(2, 2, 8), GRADIENTG2(2, 2, 8), GRADIENTF(2, 2, 8), axes = boxed)$