Start

```
restart;
with(plots);
```

[animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot] **(1.1)**

```
with(Student[MultivariateCalculus]);
```

[ApproximateInt, ApproximateIntTutor, CenterOfMass, ChangeOfVariables, CrossSection, CrossSectionTutor, Del, DirectionalDerivative, DirectionalDerivativeTutor, FunctionAverage, Gradient, GradientTutor, Jacobian, LagrangeMultipliers, MultiInt, Nabla, Revert, SecondDerivativeTest, SurfaceArea, TaylorApproximation, TaylorApproximationTutor] **(1.2)**

Eksempel 1

Finn massesentrumet til D som ligger mellom planene

x=z

```
Plan1 := implicitplot3d(x = z, x = -1 ..2, y = -2 ..2, z = 0 ..1, color = cyan, transparency = 0.1 , style = surface);
```

PLOT3D(...) **(2.1)**

z=0

```
Plan2 := implicitplot3d(z = 0, x = -1 ..2, y = -2 ..2, z = 0 ..1, color = cyan, transparency = 0, style = surface);
```

PLOT3D(...) **(2.2)**

x=1

```
Plan3 := implicitplot3d(x = 1, x = -1 ..2, y = -2 ..2, z = 0 ..1, color = cyan, transparency = 0, style = surface);
```

PLOT3D(...) **(2.3)**

og sylinderen x=y^2

```
Sylinderen := implicitplot3d(x = y^2, x = -1 ..2, y = -2 ..2, z = 0 ..1, color = gray, transparency = 0.7, style = surface);
```

PLOT3D(...) **(2.4)**

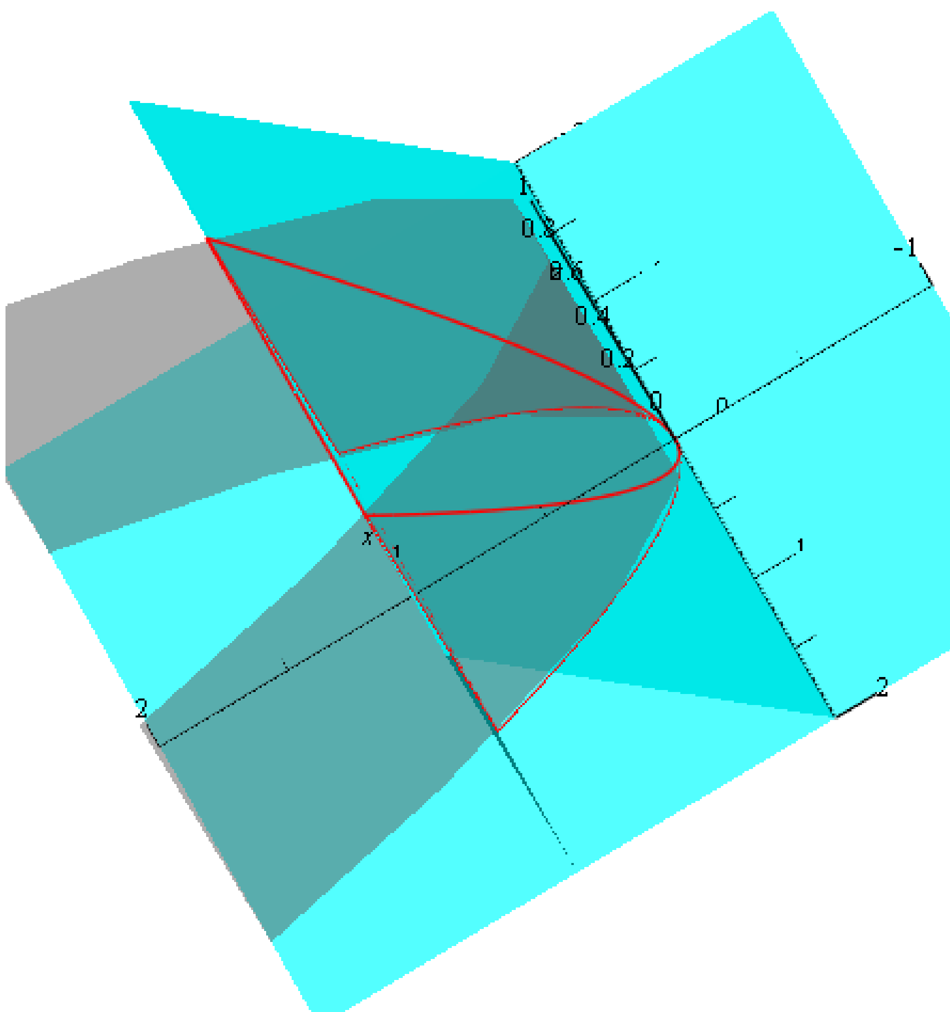
grensekurvene er

```
Grensen1 := spacecurve(⟨t^2, t, 0⟩, t = -1 ..1, color = red) :
Grensen2 := spacecurve(⟨t^2, t, t^2⟩, t = -1 ..1, color = red) :
Grensen3 := spacecurve(⟨1, t, 0⟩, t = -1 ..1, color = red) :
Grensen4 := spacecurve(⟨1, t, 1⟩, t = -1 ..1, color = red) :
```

$Grensen5 := spacecurve(\langle 1, -1, t\rangle, t = 0 ..1, color = red) :$
$Grensen6 := spacecurve(\langle 1, 1, t\rangle, t = 0 ..1, color = red) :$

D

$display(Plan1, Plan2, Plan3, Sylinderen, Grensen1, Grensen2, Grensen3, Grensen4, Grensen5, Grensen6, axes = normal, scaling = constrained)$



1
0.8
0.6
0.4
0.2
0
0
-1
x
1
2
1
2

Massetettheten er konstant d

Masse

MultiInt$(d, z = 0..x, x = y^2..1, y = -1..1, output = steps)$;

$$\int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{x} d\,dz\,dx\,dy$$

$$= \int_{-1}^{1}\int_{y^2}^{1}\left(d\,z\Big|_{z=0..x}\right)dx\,dy$$

$$= \int_{-1}^{1}\int_{y^2}^{1} d\,x\,dx\,dy$$

$$= \int_{-1}^{1}\left(\frac{d\,x^2}{2}\Bigg|_{x=y^2..1}\right)dy$$

$$= \int_{-1}^{1}\frac{d\left(1-y^4\right)}{2}\,dy$$

$$= \frac{d\left(y-\frac{1}{5}y^5\right)}{2}\Bigg|_{y=-1..1}$$

$$\frac{4}{5}d \qquad (2.5)$$

Myz

MultiInt$(d\cdot x, z = 0..x, x = y^2..1, y = -1..1, output = steps)$;

$$\int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{x} d\,x\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{-1}^{1}\int_{y^2}^{1}\left(d\,x\,z\Big|_{z=0\,..\,x}\right)\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{-1}^{1}\int_{y^2}^{1} d\,x^2\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{-1}^{1}\left(\frac{d\,x^3}{3}\Bigg|_{x=y^2\,..\,1}\right)\mathrm{d}y$$

$$= \int_{-1}^{1}\frac{d\,(1-y^6)}{3}\,\mathrm{d}y$$

$$= \frac{d\left(y-\frac{1}{7}\,y^7\right)}{3}\Bigg|_{y=-1\,..\,1}$$

$$\frac{4}{7}\,d \tag{2.6}$$

Mxz

MultiInt$\left(d\cdot y, z=0\,..x, x=y^2\,..1, y=-1\,..1, output=steps\right)$;

$$\int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{x} d\,y\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{-1}^{1}\int_{y^2}^{1}\left(d\,y\,z\Big|_{z=0\,..\,x}\right)\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{-1}^{1}\int_{y^2}^{1} d\,y\,x\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{-1}^{1}\left(\frac{d\,y\,x^2}{2}\Bigg|_{x=y^2\,..\,1}\right)\mathrm{d}y$$

$$= \int_{-1}^{1}\frac{d\,y\,(1-y^4)}{2}\,\mathrm{d}y$$

$$= \frac{d\left(-\frac{1}{6}\,y^6+\frac{1}{2}\,y^2\right)}{2}\Bigg|_{y=-1\,..\,1}$$

$$0 \qquad \textbf{(2.7)}$$

Mxy

$MultiInt\left(d\cdot z, z=0\,..x, x=y^2\,..1, y=-1\,..1, output=steps\right);$

$$\int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{x} d\,z\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{-1}^{1}\int_{y^2}^{1}\left(\left.\frac{d\,z^2}{2}\right|_{z=0\,..\,x}\right)\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{-1}^{1}\int_{y^2}^{1}\frac{d\,x^2}{2}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{-1}^{1}\left(\left.\frac{d\,x^3}{6}\right|_{x=y^2\,..\,1}\right)\mathrm{d}y$$

$$= \int_{-1}^{1}\frac{d\,(1-y^6)}{6}\,\mathrm{d}y$$

$$= \left.\frac{d\left(y-\frac{1}{7}\,y^7\right)}{6}\right|_{y=-1\,..\,1}$$

$$\frac{2}{7}\,d \qquad \textbf{(2.8)}$$

Massesentrumet

$\left\langle \frac{5}{7}, 0, \frac{5}{14} \right\rangle$

$$\begin{bmatrix} \frac{5}{7} \\ 0 \\ \frac{5}{14} \end{bmatrix} \qquad \textbf{(2.9)}$$

Eksempel 2

Skriv opp Ix, Iz, I_L til R som har massetettheten δ

δ(x,y,z)=avstand fra origo

og R ligger

under paraboloiden z=7-x^2-y^2 og

$Paraboloide := plot3d(7-x^2-y^2, x=-2\,..2, y=-2\,..2);$

$PLOT3D(...)$ **(3.1)**

inni sylinderen x^2+y^2=4 og

$Sylinderen := implicitplot3d\left(x^2 + y^2 = 4, x = -2\,..2, y = -2\,..2, z = 0\,..6, color = gray, transparency = 0.2, style = surface\right);$

$$PLOT3D(\ldots) \quad (3.2)$$

over xy-planet

$XYplanet := implicitplot3d(z = 0, x = -3\,..3, y = -3\,..3, z = 0\,..1, color = red, style = surface, transparency = 0.3);$

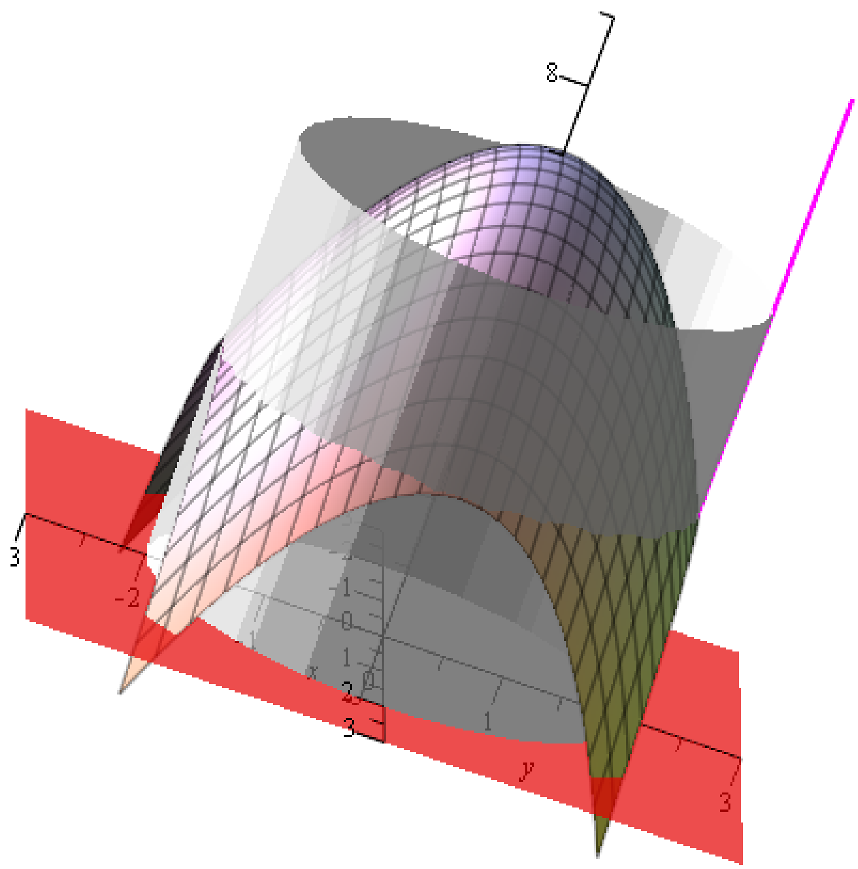
$$PLOT3D(\ldots) \quad (3.3)$$

L er parallell med z-aksen og går gjennom (0,2,0)

$L := spacecurve(\langle 0, 2, t\rangle, t = -1\,..9, color = magenta, thickness = 2);$

$$PLOT3D(\ldots) \quad (3.4)$$

$display(Paraboloide, Sylinderen, XYplanet, L, axes = normal);$



Grensene er
0 <= z <= 7-x^2-y^2
-sqrt(4 - x^2) <= y <= sqrt(4 - x^2)
-2 <= x <= 2

Massetettheten δ er

$$\sqrt{x^2+y^2+z^2}$$

Ix

$$\int_{x=-2}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}\int_{z=0}^{z=7-x^2-y^2}\left(y^2+z^2\right)\sqrt{x^2+y^2+z^2}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

Iz

$$\int_{x=-2}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}\int_{z=0}^{z=7-x^2-y^2}\left(x^2+y^2\right)\sqrt{x^2+y^2+z^2}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

I_L

$$\int_{x=-2}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}\int_{z=0}^{z=7-x^2-y^2}\left(x^2+(y-2)^2\right)\sqrt{x^2+y^2+z^2}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x=I_z+2^2\cdot M-2\,M_{xz}=I_z+2^2\cdot M$$

Massesentrumet ligger på z-aksen (på grunn av symmetri), derfor Myz = 0 og Mxz = 0
Det er et eksempel på Parallelakseteoremet (finn teoremet på internett)