Start

restart;
with(*plots*);

[*animate*, *animate3d*, *animatecurve*, *arrow*, *changecoords*, *complexplot*, *complexplot3d*, *conformal*, *conformal3d*, *contourplot*, *contourplot3d*, *coordplot*, *coordplot3d*, *densityplot*, *display*, *dualaxisplot*, *fieldplot*, *fieldplot3d*, *gradplot*, *gradplot3d*, *implicitplot*, *implicitplot3d*, *inequal*, *interactive*, *interactiveparams*, *intersectplot*, *listcontplot*, *listcontplot3d*, *listdensityplot*, *listplot*, *listplot3d*, *loglogplot*, *logplot*, *matrixplot*, *multiple*, *odeplot*, *pareto*, *plotcompare*, *pointplot*, *pointplot3d*, *polarplot*, *polygonplot*, *polygonplot3d*, *polyhedra_supported*, *polyhedraplot*, *rootlocus*, *semilogplot*, *setcolors*, *setoptions*, *setoptions3d*, *spacecurve*, *sparsematrixplot*, *surfdata*, *textplot*, *textplot3d*, *tubeplot*] **(1.1)**

with(*Student*[*MultivariateCalculus*]);

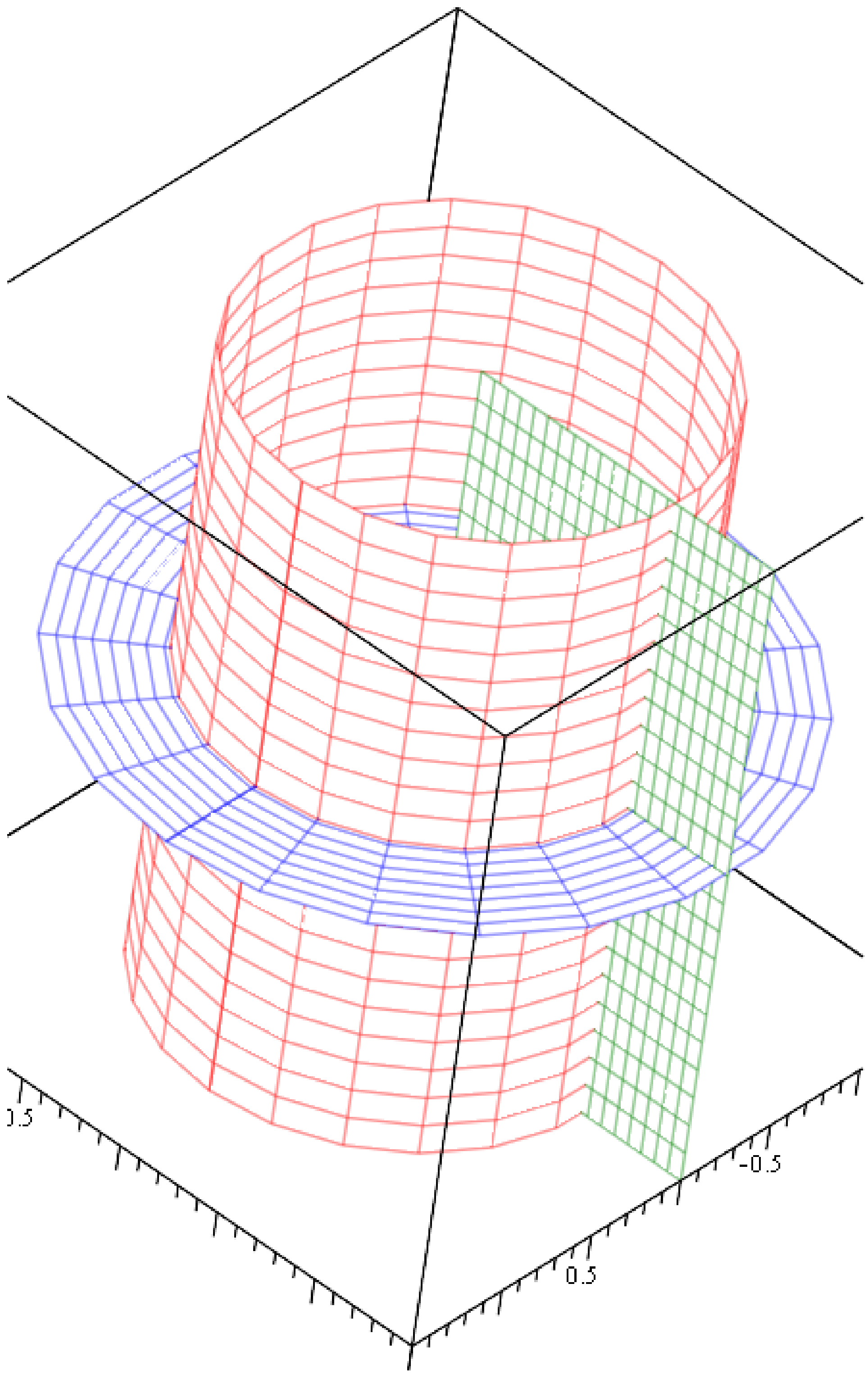
[*ApproximateInt*, *ApproximateIntTutor*, *CenterOfMass*, *ChangeOfVariables*, *CrossSection*, *CrossSectionTutor*, *Del*, *DirectionalDerivative*, *DirectionalDerivativeTutor*, *FunctionAverage*, *Gradient*, *GradientTutor*, *Jacobian*, *LagrangeMultipliers*, *MultiInt*, *Nabla*, *Revert*, *SecondDerivativeTest*, *SurfaceArea*, *TaylorApproximation*, *TaylorApproximationTutor*] **(1.2)**

with(*plottools*);

[*annulus*, *arc*, *arrow*, *circle*, *cone*, *cuboid*, *curve*, *cutin*, *cutout*, *cylinder*, *disk*, *dodecahedron*, *ellipse*, *ellipticArc*, *getdata*, *hemisphere*, *hexahedron*, *homothety*, *hyperbola*, *icosahedron*, *line*, *octahedron*, *parallelepiped*, *pieslice*, *point*, *polygon*, *prism*, *project*, *rectangle*, *reflect*, *rotate*, *scale*, *sector*, *semitorus*, *sphere*, *stellate*, *tetrahedron*, *torus*, *transform*, *translate*] **(1.3)**

Sylinderkoordinater

coordplot3d(*cylindrical*, *axes* = *boxed*);


0.5
-0.5
0.5

Eksempel 1 (Eksamen 2001 vår / 6)

Finn massesentrumet til legemet T som er sett sammen av A og B
A - en rett sirkular kjegle
- høyde 4
- grunnflateradius 2
- massetetthet 1/3

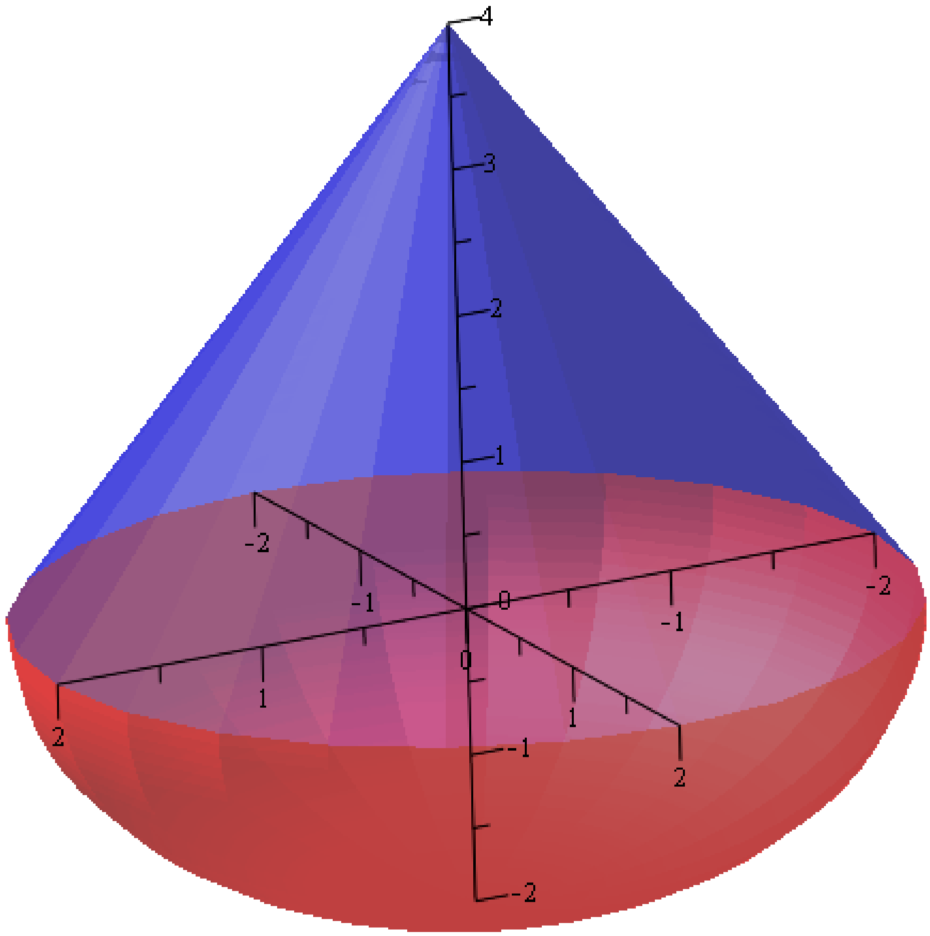
$delA := implicitplot3d(z = 4 - 2 \cdot r, r = 0 ..2, \mathrm{theta} = 0 ..2 \cdot \mathrm{Pi}, z = 0 ..4, coords = cylindrical, color = blue, style = surface, transparency = 0.5, grid = [25, 25, 25]) :$

B - en halvkule
- radius 2
- massetetthet 4/3

$delB := implicitplot3d(r^2 + z^2 = 4, r = 0 ..2, \mathrm{theta} = 0 ..2 \cdot \mathrm{Pi}, z = -2 ..0, coords = cylindrical, color = red, style = surface, transparency = 0.5, grid = [25, 25, 25]) :$

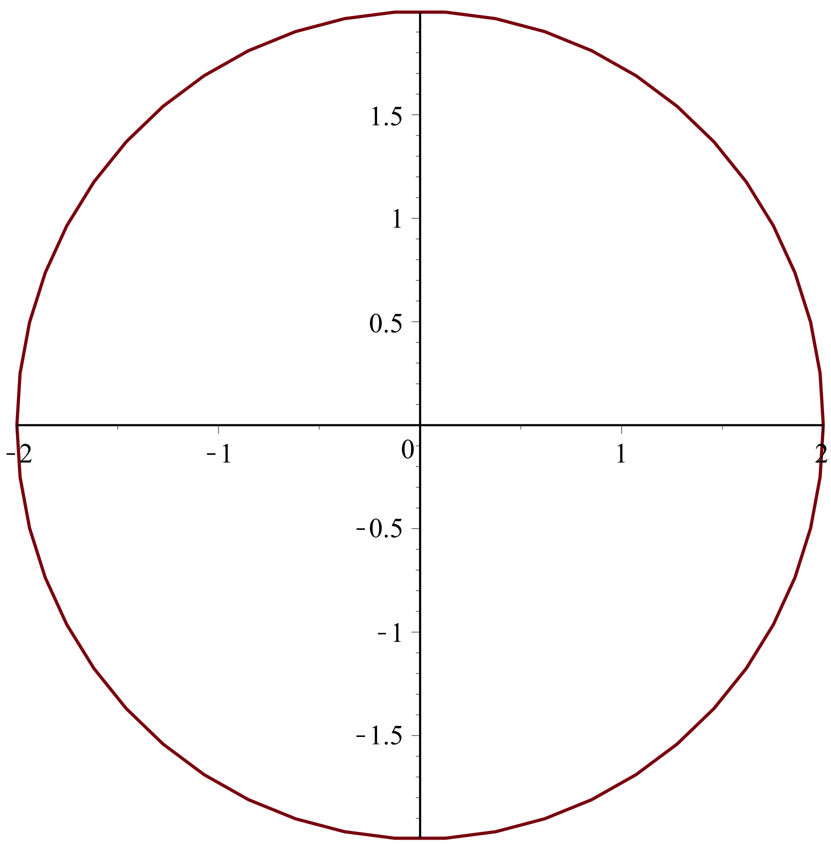
T

$display(delA, delB, axes = normal);$


4
3
2
1
0
-1
-2
-2
-1
0
1
2
-2
-1
0
1
2

Projeksjonen til T på xy-planet (samme som projeksjonen til A eller B på xy-planet)

$implicitplot\left(r=2,\ r=1..3,\ \text{theta}=0..2\cdot\pi,\ coords=\text{polar}\right);$



Projeksjonen til T på rz-planet (samme som på xz-planet, eller yz-planet) [BARE halvparten: r >= 0 nå, i stedet av r < 0 bruker vi 0 <= θ <= 2π]

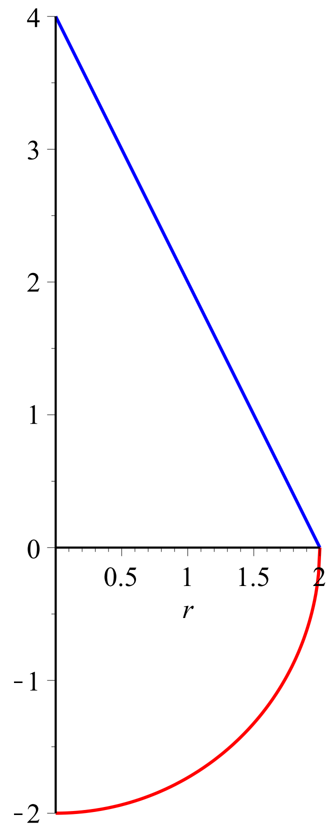
$triangel := plot(4-2\,r,\ r=0..2,\ axes=normal,\ color=blue);$

$sirkel := plot\left(-\text{sqrt}\left(4-r^2\right),\ r=0..2,\ color=red\right);$

$display(triangel,\ sirkel,\ scaling=constrained);$

$$PLOT(\ldots)$$

$$PLOT(\ldots)$$



Grensene er
A:
0 <= z <= 4 - 2r
0 <= r <= 2
0 <= θ <= 2π

B:
-sqrt(4-r^2) <= z <= 0
0 <= r <= 2
0 <= θ <= 2π

T er symmetrisk om z-aksen -> Massesentrumet = < 0, 0, Mxy / M >
M = Ma + Mb
Ma

$$\mathit{MultiInt}\left(\frac{1}{3}\cdot r, z = 0 ..4 - 2\cdot r, r = 0 ..2, \text{theta} = 0 ..2\cdot \text{Pi}, \mathit{output} = \mathit{steps}\right);$$

$$\int_0^{2\pi}\int_0^2\int_0^{4-2r} \frac{r}{3}\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \left(\left. \frac{r z}{3} \right|_{z=0..4-2r} \right) dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \frac{r(4-2r)}{3}\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(\left. \left(-\frac{2}{9} r^3 + \frac{2}{3} r^2 \right) \right|_{r=0..2} \right) d\theta$$

$$= \int_0^{2\pi} \frac{8}{9}\, d\theta$$

$$= \left. \frac{8\,\theta}{9} \right|_{\theta=0..2\pi}$$

$$\frac{16}{9}\pi \tag{3.1}$$

Mb

$MultiInt\left(\frac{4}{3} \cdot r, z = -\mathrm{sqrt}\left(4 - r^2\right) ..0, r = 0..2, \mathrm{theta} = 0..2 \cdot \mathrm{Pi}, output = steps \right);$

$$\int_0^{2\pi}\int_0^{2}\int_{-\sqrt{4-r^2}}^{0} \frac{4\,r}{3}\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} \left(\left. \frac{4\,r\,z}{3} \right|_{z=-\sqrt{4-r^2}\,..0} \right) dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} \frac{4\,r\sqrt{4-r^2}}{3}\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\left. \frac{4\,(-2+r)\,(r+2)\,\sqrt{4-r^2}}{9} \right|_{r=0\,..2} \right) d\theta$$

$$= \int_0^{2\pi} \frac{32}{9}\,d\theta$$

$$= \left. \frac{32\,\theta}{9} \right|_{\theta=0\,..2\,\pi}$$

$$\frac{64}{9}\,\pi \qquad \textbf{(3.2)}$$

Massen er

$$\frac{16}{9}\,\pi + \frac{64}{9}\,\pi$$

$$\frac{80}{9}\,\pi \qquad \textbf{(3.3)}$$

Mxy = MxyA + MxyB
MxyA

$MultiInt\left(\frac{1}{3}\cdot z\cdot r,\ z=0\,..4-2\cdot r,\ r=0\,..2,\ \text{theta}=0\,..2\cdot\text{Pi},\ output=steps\right);$

$$\int_0^{2\pi}\int_0^2\int_0^{4-2r} \frac{r\,z}{3}\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \left(\left. \frac{r\,z^2}{6} \right|_{z=0..4-2r} \right) dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \frac{r\,(4-2\,r)^2}{6}\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\left. \left(\frac{1}{6}\,r^4 - \frac{8}{9}\,r^3 + \frac{4}{3}\,r^2 \right) \right|_{r=0..2} \right) d\theta$$

$$= \int_0^{2\pi} \frac{8}{9}\,d\theta$$

$$= \left. \frac{8\,\theta}{9} \right|_{\theta=0..2\pi}$$

$$\frac{16}{9}\,\pi \qquad \textbf{(3.4)}$$

MxyB

$MultiInt\left(\frac{4}{3}\cdot z\cdot r,\ z=-\mathrm{sqrt}\left(4-r^2\right)..0,\ r=0..2,\ \mathrm{theta}=0..2\cdot\mathrm{Pi},\ output=steps\right);$

$$\int_0^{2\pi}\int_0^2\int_{-\sqrt{4-r^2}}^0 \frac{4\,r\,z}{3}\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \left(\left. \frac{2\,r\,z^2}{3} \right|_{z=-\sqrt{4-r^2}\,..0} \right) dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \frac{2\,(-4+r^2)\,r}{3}\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\left. \left(\frac{1}{6}\,r^4 - \frac{4}{3}\,r^2 \right) \right|_{r=0\,..2} \right) d\theta$$

$$= \int_0^{2\pi} -\frac{8}{3}\,d\theta$$

$$= \left. -\frac{8\,\theta}{3} \right|_{\theta=0\,..2\,\pi}$$

$$-\frac{16}{3}\,\pi \tag{3.5}$$

Mxy

$\frac{16}{9}\,\pi - \frac{16}{3}\,\pi$

$$-\frac{32}{9}\,\pi \tag{3.6}$$

Mxy / M

$\frac{\frac{16}{9}\,\pi - \frac{16}{3}\,\pi}{\frac{80}{9}\,\pi}$

$$-\frac{2}{5} \tag{3.7}$$

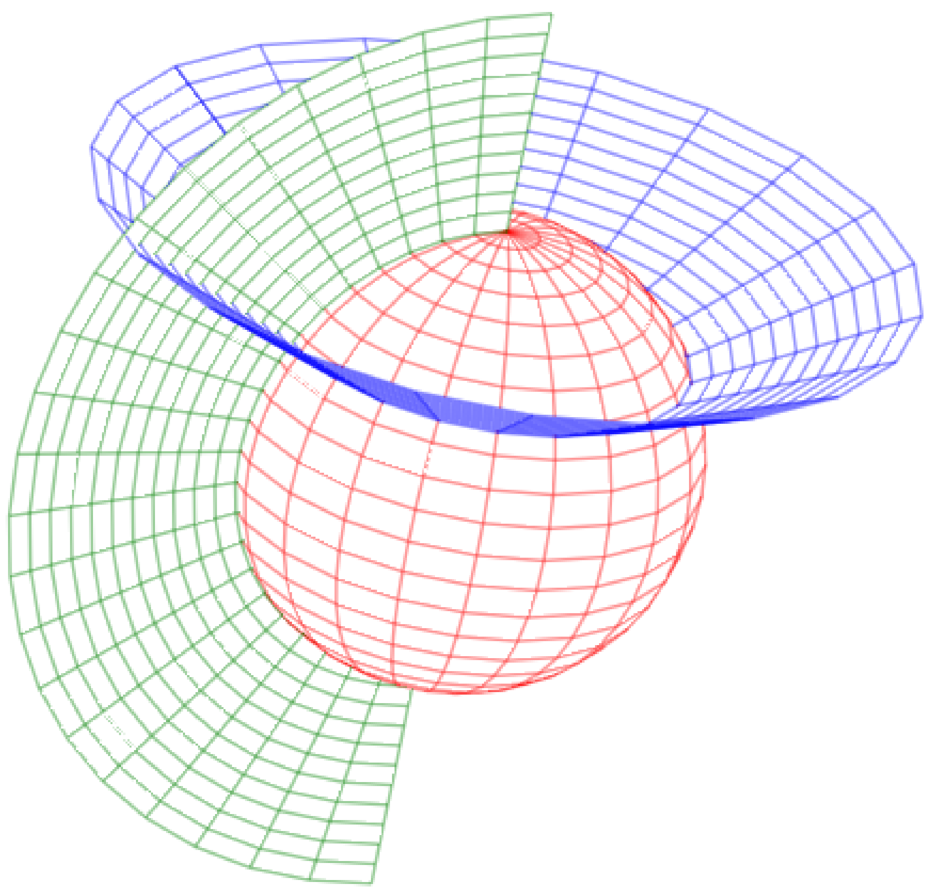
T er symmetrisk om z-aksen, derfor er massesentrumet <0, 0, Mxy / M>

$\left\langle 0, 0, -\frac{2}{5} \right\rangle$

$$\begin{bmatrix} 0 \\ 0 \\ -\frac{2}{5} \end{bmatrix} \tag{3.8}$$

Kulekoordinater

coordplot3d(*spherical*);



Eksempel 2

Finn massesen til R som ligger mellom
- sfæren med radius 2

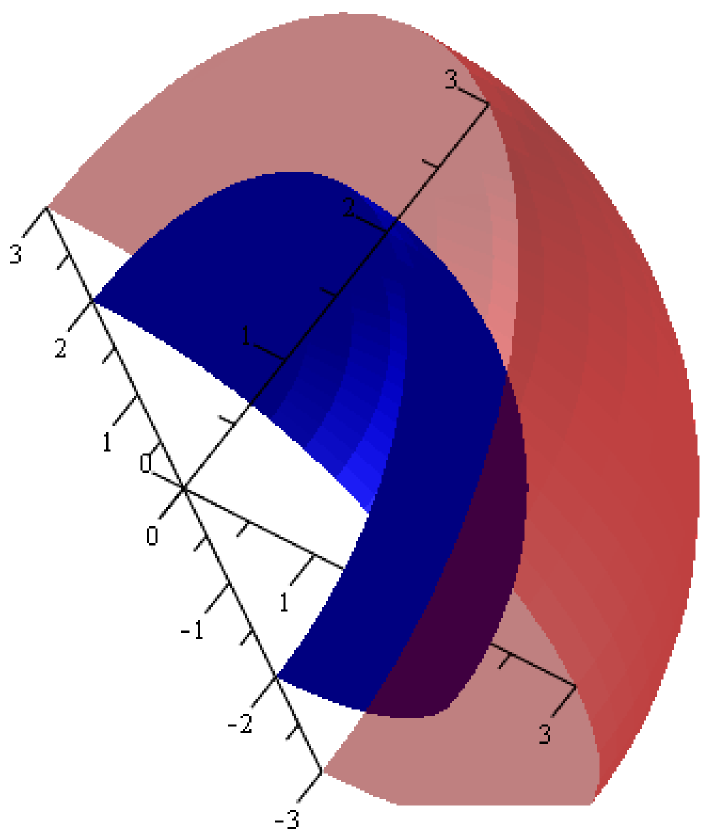
$sfaere1 := implicitplot3d\left(\mathrm{rho}=2, \mathrm{rho}=0..4, \mathrm{phi}=0..\frac{\mathrm{Pi}}{2}, \mathrm{theta}=0..\mathrm{Pi}, coords=spherical, color=blue, style=surface, grid=[25, 25, 25]\right):$

- og sfæren med radius 3

$sfaere2 := implicitplot3d\left(\mathrm{rho}=3, \mathrm{rho}=0..4, \mathrm{phi}=0..\frac{\mathrm{Pi}}{2}, \mathrm{theta}=0..\mathrm{Pi}, coords=spherical, color=\mathrm{red}, \mathrm{transparency}=0.5, style=surface, grid=[25, 25, 25]\right):$

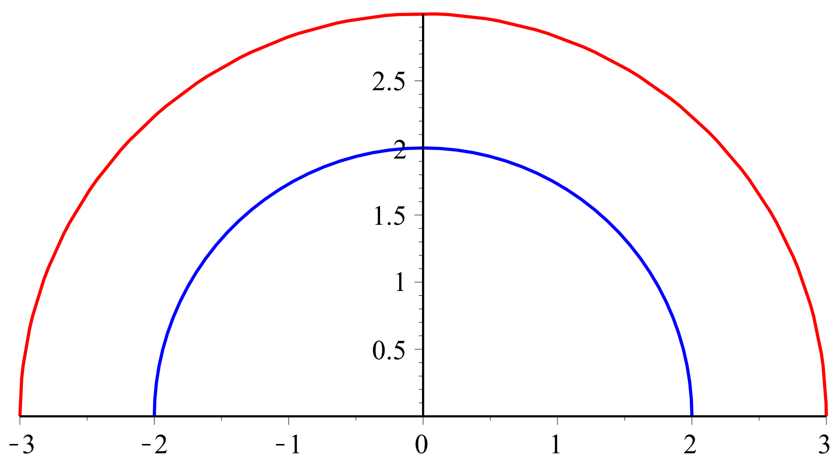
R

$display(sfaere1, sfaere2, axes=normal, scaling=constrained);$



- projeksjonen til R på xy-planet

$implicitplot([r=2, r=3], r=0..4, \mathrm{theta}=0..\mathrm{Pi}, coords=\mathrm{polar}, color=[blue, red], axes=normal, scaling=constrained);$



- massetettheten = avstand fra origo (δ = ρ)

Finn grensene
2 <= ρ <= 3
0 <= Φ<= π/2
0 <= θ <= π

Massen

$$MultiInt\left(\rho\cdot\rho^{2}\cdot\sin(\phi),\ \rho=2\,..3,\ \phi=0\,..\frac{\pi}{2},\ \theta=0\,..\pi,\ output=steps\right);$$

$$\int_0^{\pi}\int_0^{\frac{\pi}{2}}\int_2^{3}\rho^3\sin(\phi)\,d\rho\,d\phi\,d\theta$$

$$=\int_0^{\pi}\int_0^{\frac{\pi}{2}}\left(\left.\frac{\rho^4\sin(\phi)}{4}\right|_{\rho=2..3}\right)d\phi\,d\theta$$

$$=\int_0^{\pi}\int_0^{\frac{\pi}{2}}\frac{65\sin(\phi)}{4}\,d\phi\,d\theta$$

$$=\int_0^{\pi}\left(\left.-\frac{65\cos(\phi)}{4}\right|_{\phi=0..\frac{\pi}{2}}\right)d\theta$$

$$=\int_0^{\pi}\frac{65}{4}\,d\theta$$

$$=\left.\frac{65\,\theta}{4}\right|_{\theta=0..\pi}$$

$$\frac{65}{4}\pi \qquad \textbf{(5.1)}$$

Eksempel 3: Integrasjon ved substitusjon (2D)

f er gitt ved ((x-y)/(x+y+2))^2
R er gitt ved

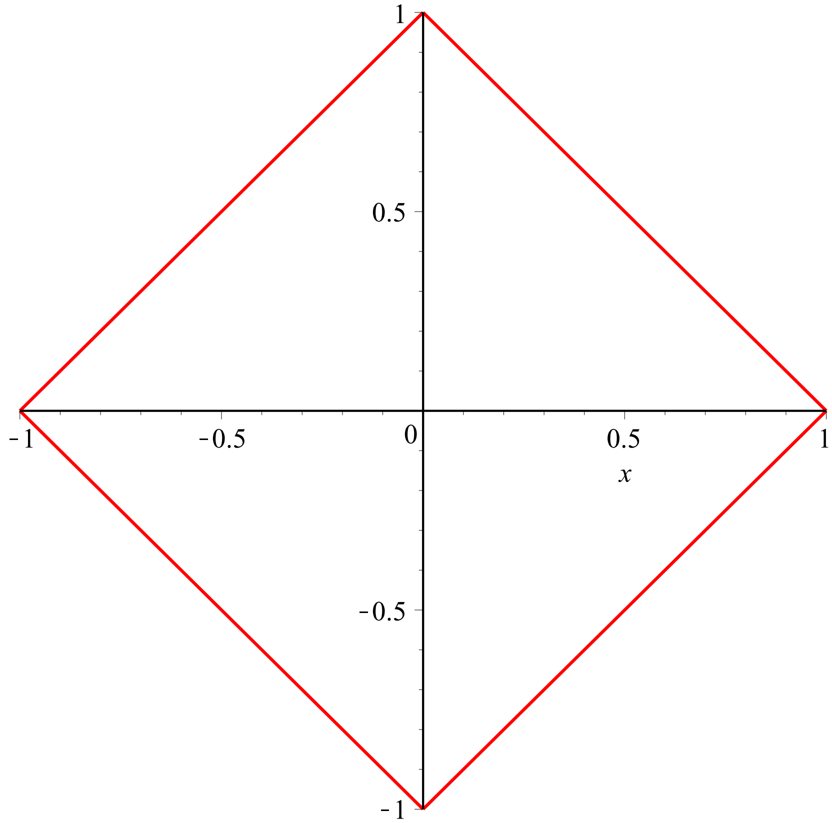
$del1 := plot([-1-x, 1+x], x=-1..0, color=[red, red]);$

$$PLOT(\ldots) \qquad \textbf{(6.1)}$$

$del2 := plot([1-x, -1+x], x=0..1, color=[red, red]);$

$$PLOT(\ldots) \qquad \textbf{(6.2)}$$

$display(del1, del2);$



Innfør nye variable
u = x + y
v = x - y
og finn integralet til f over R

Vi finner x og y som funksjoner av u og v: x = g(u,v) og y = h(u,v)

$g := (u, v) \to \dfrac{u+v}{2};$

$h := (u, v) \to \dfrac{u-v}{2};$

$$(u, v) \to \frac{1}{2}\, u + \frac{1}{2}\, v$$

$$(u, v) \to \frac{1}{2}\, u - \frac{1}{2}\, v \qquad \textbf{(6.3)}$$

Jacobi matrisen

$Jacobian([g(u, v), h(u, v)], [u, v]);$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \quad \textbf{(6.4)}$$

Jacobideterminanten er

$\frac{1}{2} \cdot \frac{-1}{2} - \frac{1}{2} \cdot \frac{1}{2};$

$$-\frac{1}{2} \quad \textbf{(6.5)}$$

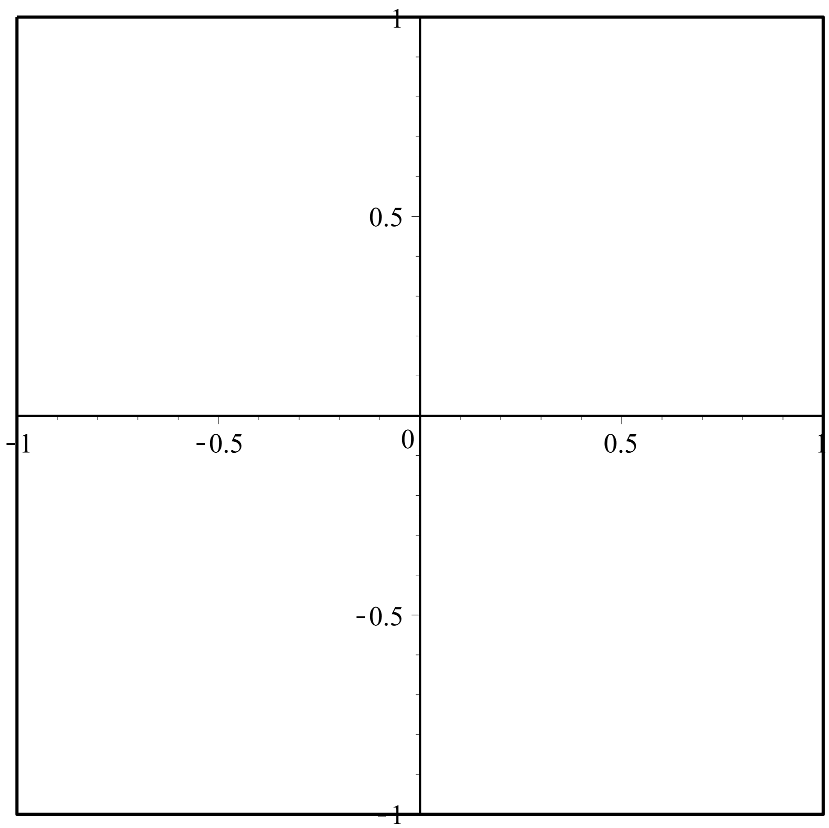
Absoluttverdien til Jacobideterminanten er

$\frac{1}{2}$

$$\frac{1}{2} \quad \textbf{(6.6)}$$

R er transformert til S

$display(rectangle([-1, 1], [1, -1]), style = line);$



Integralet i de nye koordinatene er

$$MultiInt\left(\left(\frac{v}{u+2}\right)^2\cdot\frac{1}{2}, u=-1..1, v=-1..1, output=steps\right);$$

$$\int_{-1}^{1}\int_{-1}^{1}\frac{v^2}{2\,(u+2)^2}\,\mathrm{d}u\,\mathrm{d}v$$

$$=\int_{-1}^{1}\left(-\frac{v^2}{2\,(u+2)}\Bigg|_{u=-1..1}\right)\mathrm{d}v$$

$$=\int_{-1}^{1}\frac{v^2}{3}\,\mathrm{d}v$$

$$=\frac{v^3}{9}\Bigg|_{v=-1..1}$$

$$\frac{2}{9} \quad \textbf{(6.7)}$$

Eksempel 4: Integrasjon ved substitusjon (2D)

Finn arealet til R som grenset av
to parabler y=sqrt(2x), y=sqrt(3x)

$del1 := plot([\mathrm{sqrt}(2\cdot x), \mathrm{sqrt}(3\cdot x)], x=0..4);$

$$PLOT(...) \quad \textbf{(7.1)}$$

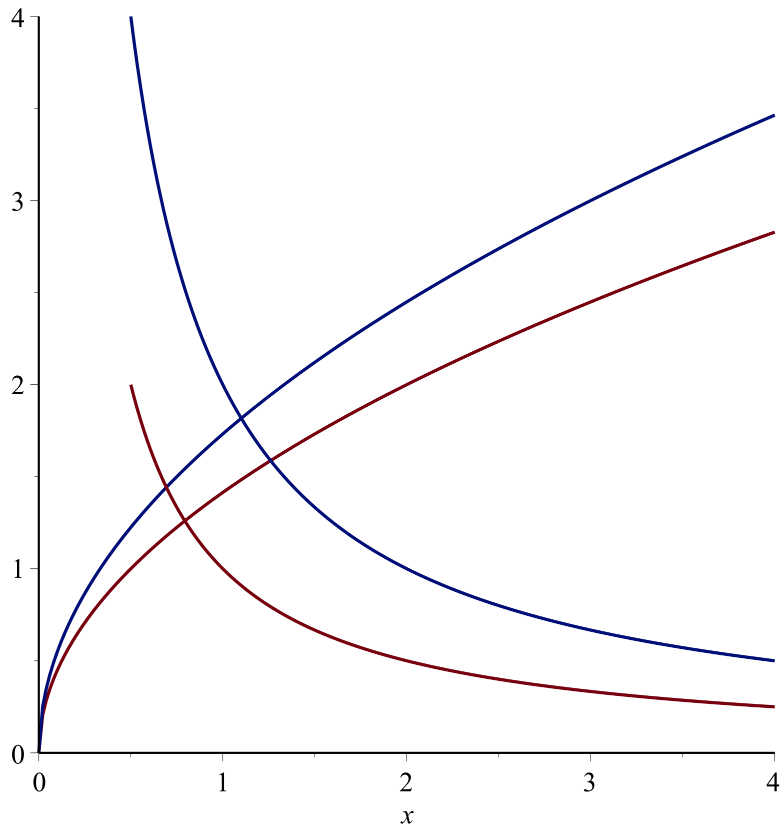
og to hyperbler y=1/x, y=2/x

$del2 := plot\left(\left[\frac{1}{x}, \frac{2}{x}\right], x=0.5..4\right);$

$$PLOT(...) \quad \textbf{(7.2)}$$

R

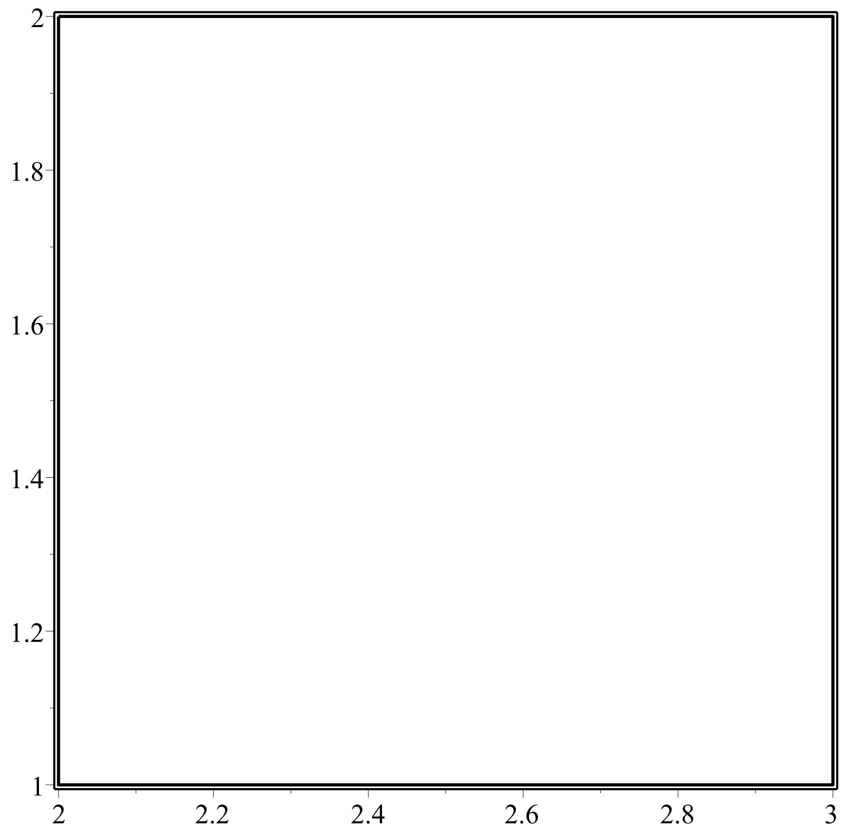
$display(del1, del2, scaling=constrained);$



Innfør nye variablene
u = y^2 / x
v = xy

Parabelen y = sqrt(2x) blir u = 2
Parabelen y = sqrt(3x) blir u = 3
Hyperbelen y=1/x blir v = 1
Hyperbelen y=2/x blir v = 2

Derfor er R transformert til rektangelen S
$display(rectangle([2, 2], [3, 1]), style = line, axes = boxed);$



Vi vil beregne arealet -> f = 1

$$\iint_R 1 \, dx \, dy$$

f = 1 også i (u,v) koordinater

Vi må bare finne absoluttverdien til Jacobideterminanten
Husk at det[∂(x,y)/∂(u,v)] = 1 / det[∂(u,v)/∂(x,y)]

∂(u,v)/∂(x,y)

$$Jacobian\left(\left[\frac{y^2}{x}, x \cdot y\right], [x, y]\right);$$

$$\begin{bmatrix} -\frac{y^2}{x^2} & \frac{2y}{x} \\ y & x \end{bmatrix} \qquad (7.3)$$

determinanten til ∂(u,v)/∂(x,y) er

$$-\frac{y^2}{x^2} \cdot x - y \cdot \frac{2 \cdot y}{x};$$

$$-\frac{3\,y^2}{x} \tag{7.4}$$

Derfor er determinanten til ∂(x,y)/∂(u,v)

$-\frac{x}{3\cdot y^2};$

$$-\frac{1}{3}\,\frac{x}{y^2} \tag{7.5}$$

Absoluttverdien til det[∂(x,y)/∂(u,v)] er

$\frac{1}{3\cdot u};$

$$\frac{1}{3\,u} \tag{7.6}$$

Vi får integralet ved substitusjon (og det er arelet til R)

$MultiInt\left(1\cdot\frac{1}{3\cdot u},\ u=2..3,\ v=1..2,\ output=steps\right);$

$$\int_1^2\int_2^3 \frac{1}{3\,u}\,du\,dv$$

$$=\int_1^2 \left(\left.\frac{\ln(u)}{3}\right|_{u=2..3}\right)dv$$

$$=\int_1^2 \left(-\frac{\ln(2)}{3}+\frac{\ln(3)}{3}\right)dv$$

$$=\left.\left(-\frac{\ln(2)}{3}+\frac{\ln(3)}{3}\right)v\right|_{v=1..2}$$

$$-\frac{1}{3}\ln(2)+\frac{1}{3}\ln(3) \tag{7.7}$$