Start

restart;
with(*plots*);

[*animate*, *animate3d*, *animatecurve*, *arrow*, *changecoords*, *complexplot*, *complexplot3d*, *conformal*, *conformal3d*, *contourplot*, *contourplot3d*, *coordplot*, *coordplot3d*, *densityplot*, *display*, *dualaxisplot*, *fieldplot*, *fieldplot3d*, *gradplot*, *gradplot3d*, *implicitplot*, *implicitplot3d*, *inequal*, *interactive*, *interactiveparams*, *intersectplot*, *listcontplot*, *listcontplot3d*, *listdensityplot*, *listplot*, *listplot3d*, *loglogplot*, *logplot*, *matrixplot*, *multiple*, *odeplot*, *pareto*, *plotcompare*, *pointplot*, *pointplot3d*, *polarplot*, *polygonplot*, *polygonplot3d*, *polyhedra_supported*, *polyhedraplot*, *rootlocus*, *semilogplot*, *setcolors*, *setoptions*, *setoptions3d*, *spacecurve*, *sparsematrixplot*, *surfdata*, *textplot*, *textplot3d*, *tubeplot*] **(1.1)**

with(*Student*[*VectorCalculus*]);

[*&x*, `*`, `+`, `-`, `.`, <,>, <|>, *About*, *ArcLength*, *BasisFormat*, *Binormal*, *ConvertVector*, *CrossProduct*, *Curl*, *Curvature*, D, *Del*, *DirectionalDiff*, *Divergence*, *DotProduct*, *FlowLine*, *Flux*, *GetCoordinates*, *GetPVDescription*, *GetRootPoint*, *GetSpace*, *Gradient*, *Hessian*, *IsPositionVector*, *IsRootedVector*, *IsVectorField*, *Jacobian*, *Laplacian*, *LineInt*, *MapToBasis*, *Nabla*, *Norm*, *Normalize*, *PathInt*, *PlotPositionVector*, *PlotVector*, *PositionVector*, *PrincipalNormal*, *RadiusOfCurvature*, *RootedVector*, *ScalarPotential*, *SetCoordinates*, *SpaceCurve*, *SpaceCurveTutor*, *SurfaceInt*, *TNBFrame*, *Tangent*, *TangentLine*, *TangentPlane*, *TangentVector*, *Torsion*, *Vector*, *VectorField*, *VectorFieldTutor*, *VectorPotential*, *VectorSpace*, *diff*, *evalVF*, *int*, *limit*, *series*] **(1.2)**

with(*Student*[*MultivariateCalculus*]);

[*ApproximateInt*, *ApproximateIntTutor*, *CenterOfMass*, *ChangeOfVariables*, *CrossSection*, *CrossSectionTutor*, *Del*, *DirectionalDerivative*, *DirectionalDerivativeTutor*, *FunctionAverage*, *Gradient*, *GradientTutor*, *Jacobian*, *LagrangeMultipliers*, *MultiInt*, *Nabla*, *Revert*, *SecondDerivativeTest*, *SurfaceArea*, *TaylorApproximation*, *TaylorApproximationTutor*] **(1.3)**

with(*Student*[*LinearAlgebra*]);

[*&x*, `.`, *AddRow*, *AddRows*, *Adjoint*, *ApplyLinearTransformPlot*, *BackwardSubstitute*, *BandMatrix*, *Basis*, *BilinearForm*, *CharacteristicMatrix*, *CharacteristicPolynomial*, *ColumnDimension*, *ColumnSpace*, *CompanionMatrix*, *ConstantMatrix*, *ConstantVector*, *CrossProductPlot*, *Determinant*, *Diagonal*, *DiagonalMatrix*, *Dimension*, *Dimensions*, *EigenPlot*, *EigenPlotTutor*, *Eigenvalues*, *EigenvaluesTutor*, *Eigenvectors*, *EigenvectorsTutor*, *Equal*, *GaussJordanEliminationTutor*, *GaussianElimination*, *GaussianEliminationTutor*, *GenerateEquations*, *GenerateMatrix*, *GramSchmidt*, *HermitianTranspose*, *Id*, *IdentityMatrix*, *IntersectionBasis*, *InverseTutor*, *IsDefinite*, *IsOrthogonal*, *IsSimilar*, *IsUnitary*, *JordanBlockMatrix*, *JordanForm*, *LUDecomposition*, *LeastSquares*, *LeastSquaresPlot*, *LinearSolve*, *LinearSolveTutor*, *LinearSystemPlot*, *LinearSystemPlotTutor*, *LinearTransformPlot*, *LinearTransformPlotTutor*, *MatrixBuilder*, *MinimalPolynomial*, *Minor*, *MultiplyRow*, *Norm*, *Normalize*, *NullSpace*, *Pivot*, *PlanePlot*, *ProjectionPlot*, *QRDecomposition*, *RandomMatrix*, *RandomVector*, **(1.4)**

Rank, *ReducedRowEchelonForm*, *ReflectionMatrix*, *RotationMatrix*, *RowDimension*, *RowSpace*, *SetDefault*, *SetDefaults*, *SumBasis*, *SwapRow*, *SwapRows*, *Trace*, *Transpose*, *UnitVector*, *VectorAngle*, *VectorSumPlot*, *ZeroMatrix*, *ZeroVector*]

with(*Student*[*Calculus1*]);

[*AntiderivativePlot*, *AntiderivativeTutor*, *ApproximateInt*, *ApproximateIntTutor*, *ArcLength*, *ArcLengthTutor*, *Asymptotes*, *Clear*, *CriticalPoints*, *CurveAnalysisTutor*, *DerivativePlot*, *DerivativeTutor*, *DiffTutor*, *ExtremePoints*, *FunctionAverage*, *FunctionAverageTutor*, *FunctionChart*, *FunctionPlot*, *GetMessage*, *GetNumProblems*, *GetProblem*, *Hint*, *InflectionPoints*, *IntTutor*, *Integrand*, *InversePlot*, *InverseTutor*, *LimitTutor*, *MeanValueTheorem*, *MeanValueTheoremTutor*, *NewtonQuotient*, *NewtonsMethod*, *NewtonsMethodTutor*, *PointInterpolation*, *RiemannSum*, *RollesTheorem*, *Roots*, *Rule*, *Show*, *ShowIncomplete*, *ShowSolution*, *ShowSteps*, *Summand*, *SurfaceOfRevolution*, *SurfaceOfRevolutionTutor*, *Tangent*, *TangentSecantTutor*, *TangentTutor*, *TaylorApproximation*, *TaylorApproximationTutor*, *Understand*, *Undo*, *VolumeOfRevolution*, *VolumeOfRevolutionTutor*, *WhatProblem*] **(1.5)**

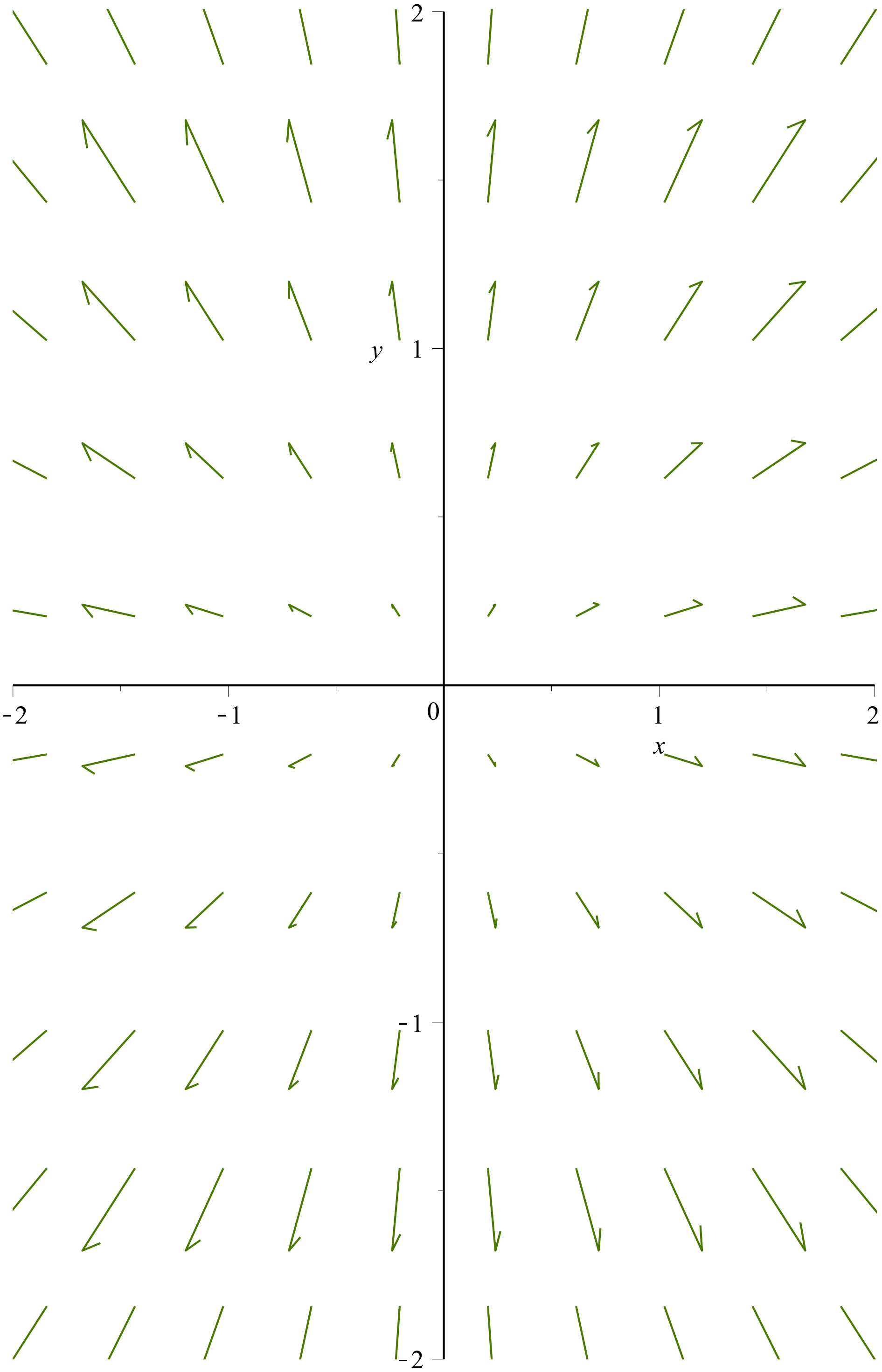
Radialfelter (radial fields)

(for eksempel: gravitasjonsfelt, elektrisk felt)

i 2D:

R_1, |R_1| = r

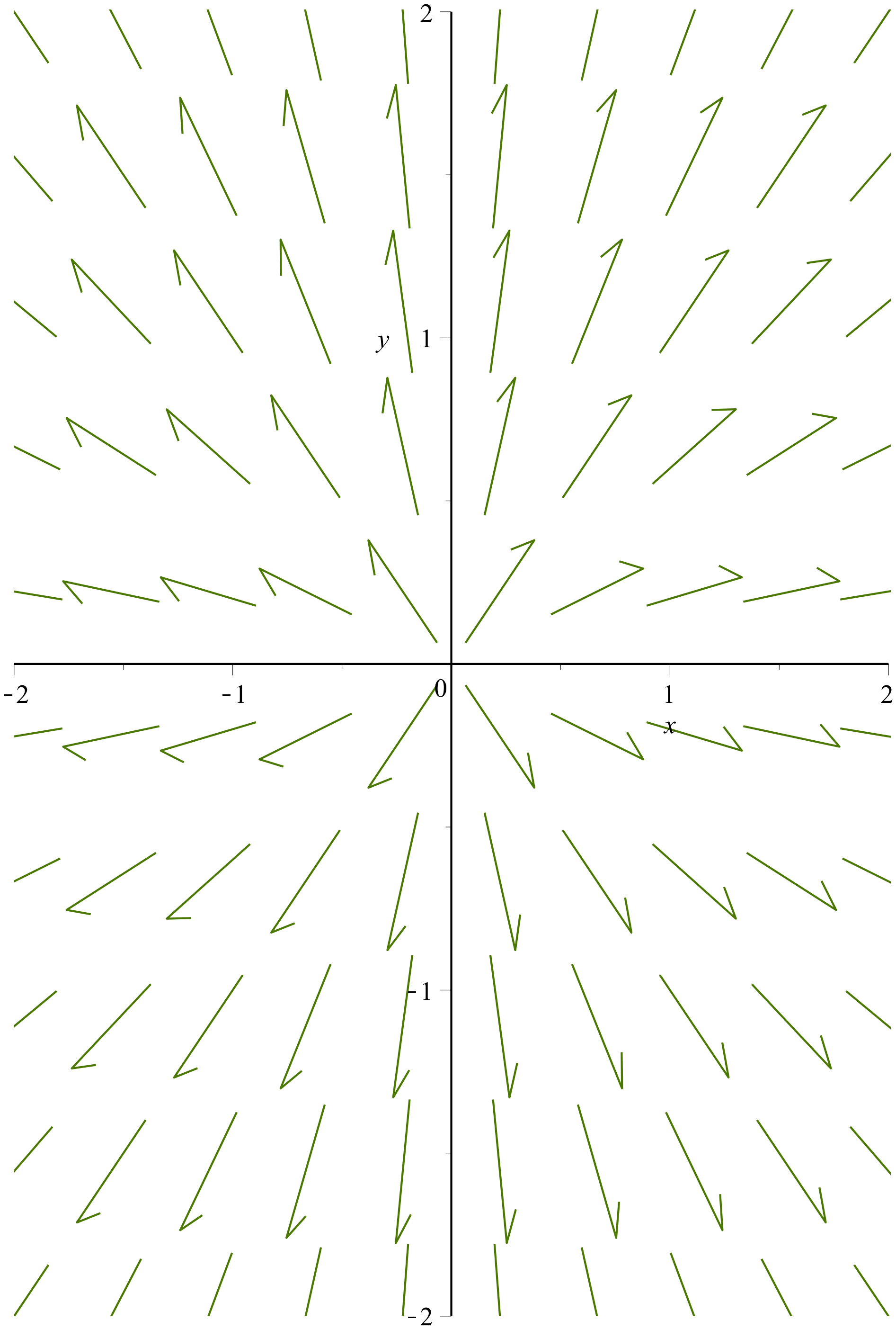
VectorField($\langle x, y\rangle$, *cartesian*[x, y], output = plot, fieldoptions = [*grid* = [10, 10]], *axes* = *normal*);



2
1
y
-2
-1
0
1
2
x
-1
-2

R_2, |R_2| = 1

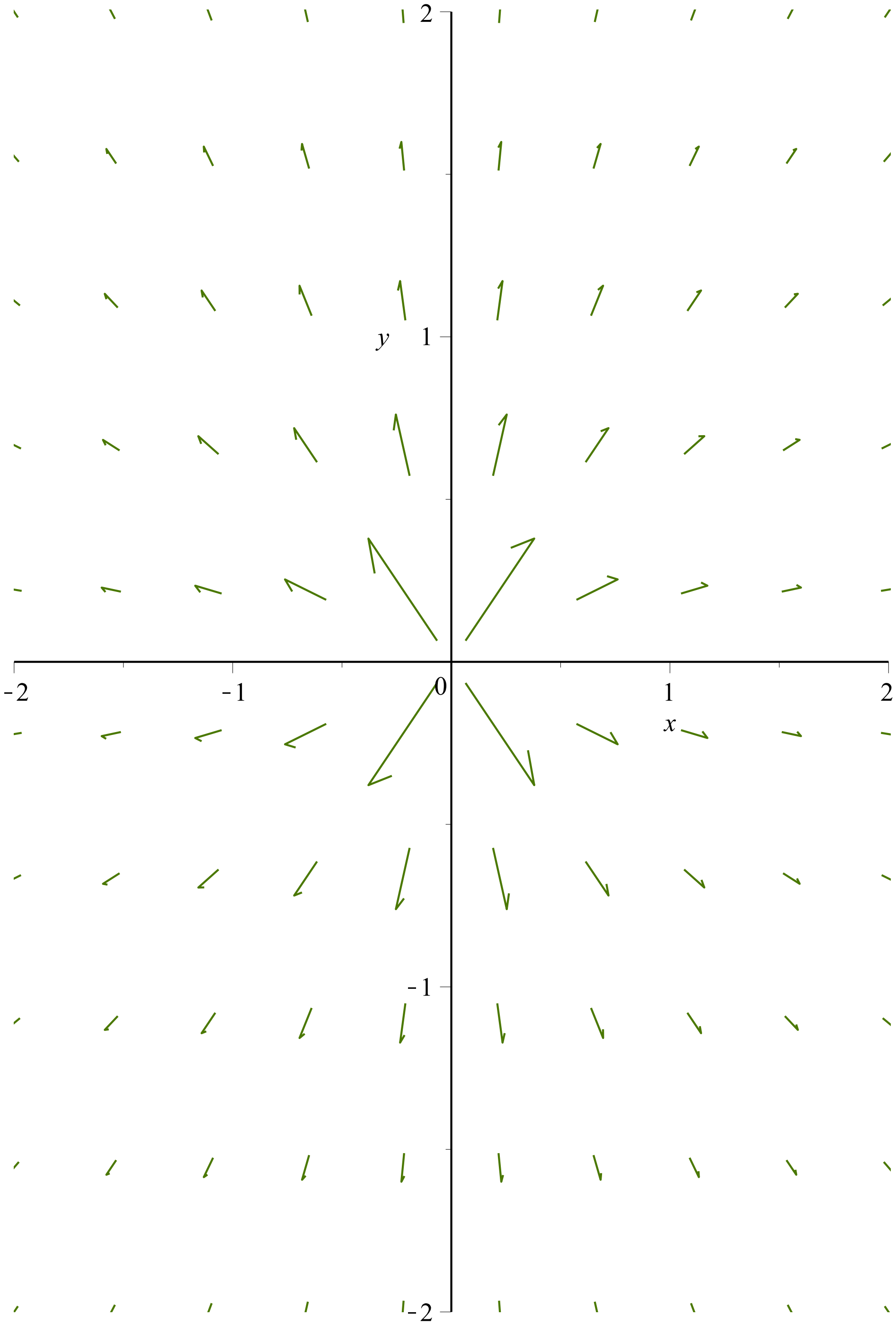
$VectorField\left(\frac{\langle x, y\rangle}{\mathrm{sqrt}\left(x^2+y^2\right)}, cartesian[x, y], \mathrm{output}=\mathrm{plot}, \mathrm{fieldoptions}=[grid=[10, 10]], axes=normal\right);$



2
1
y
-2
-1
0
1
2
x
-1
-2

R_3, |R_3| = 1/r

$VectorField\left(\frac{\langle x, y\rangle}{x^2+y^2}, cartesian[x, y], \text{output} = \text{plot}, \text{fieldoptions} = [grid = [10, 10]], axes = normal \right);$

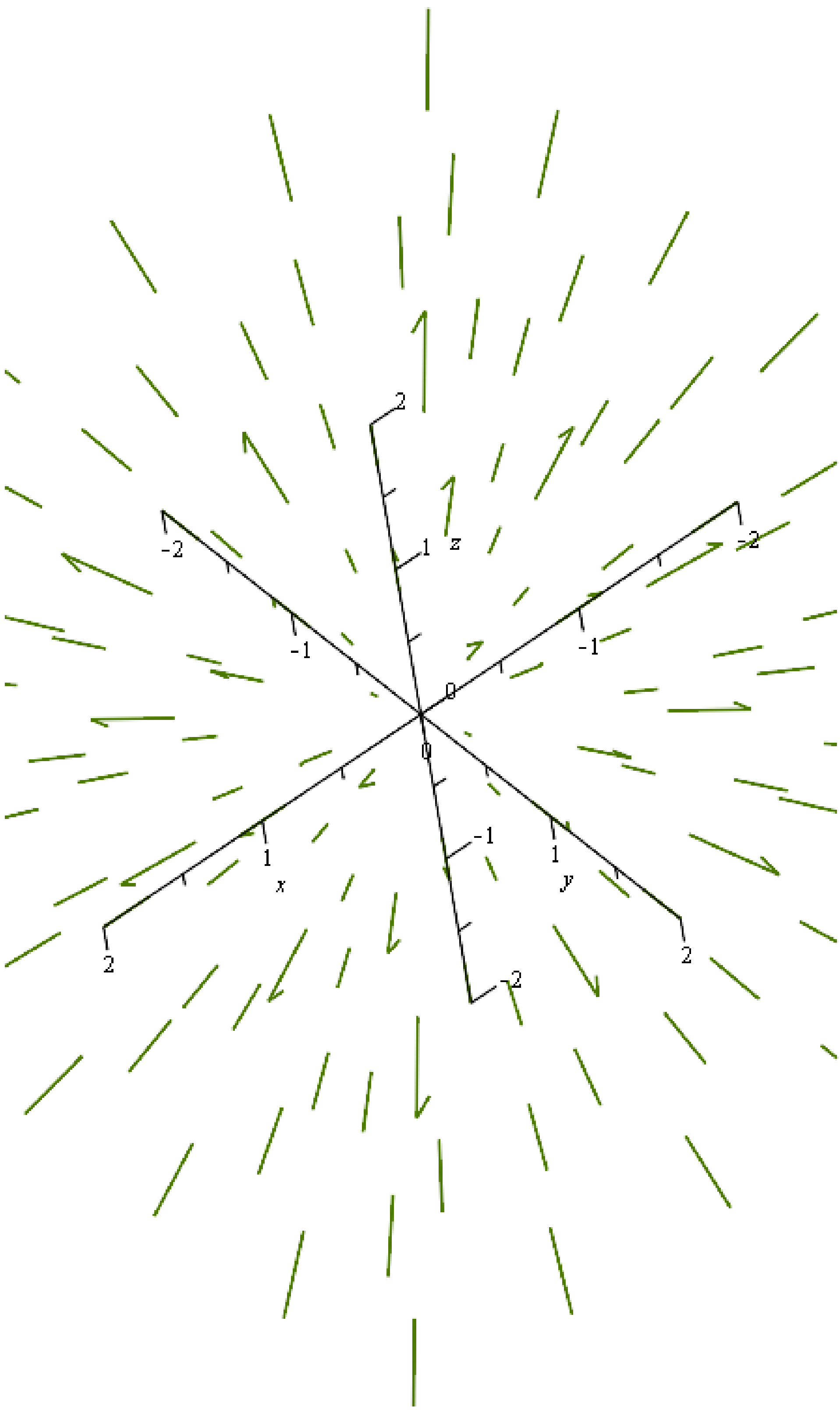


y
x
2
1
-1
-2
-2
-1
0
1
2

i 3D

R3d_1, |R3d_1| = r

VectorField(⟨*x*, *y*, *z*⟩ , *cartesian*[*x*, *y*, *z*], output = plot, fieldoptions = [*grid* = [5, 5, 5]], *axes* = *normal*);



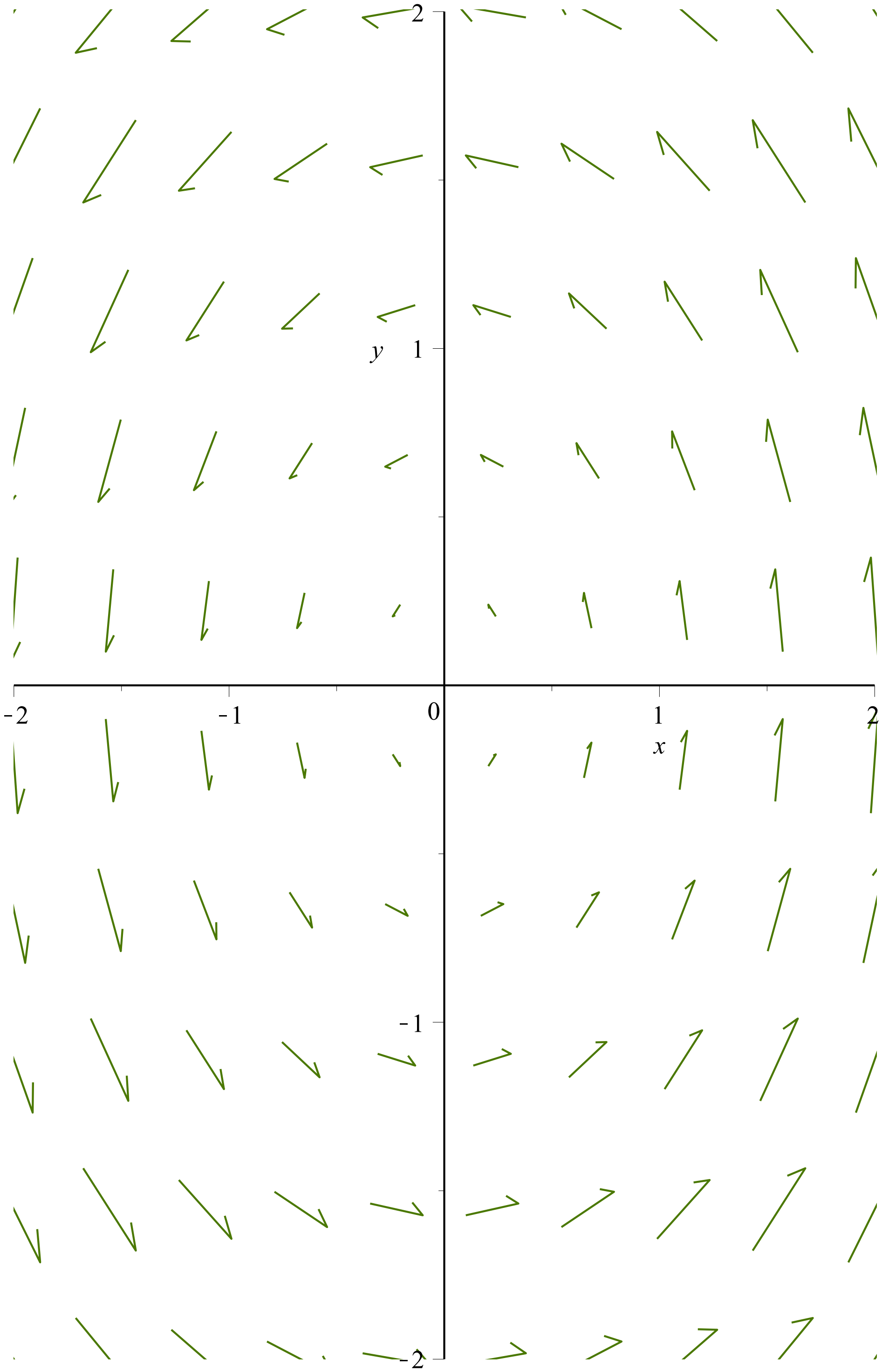
2
1
z
-2
-1
0
-1
-2
0
-1
-2
1
x
1
y
2
2

Rotasjonsfelter (spin fields)

(for eksempel en orkan)

S_1, |S_1| = r

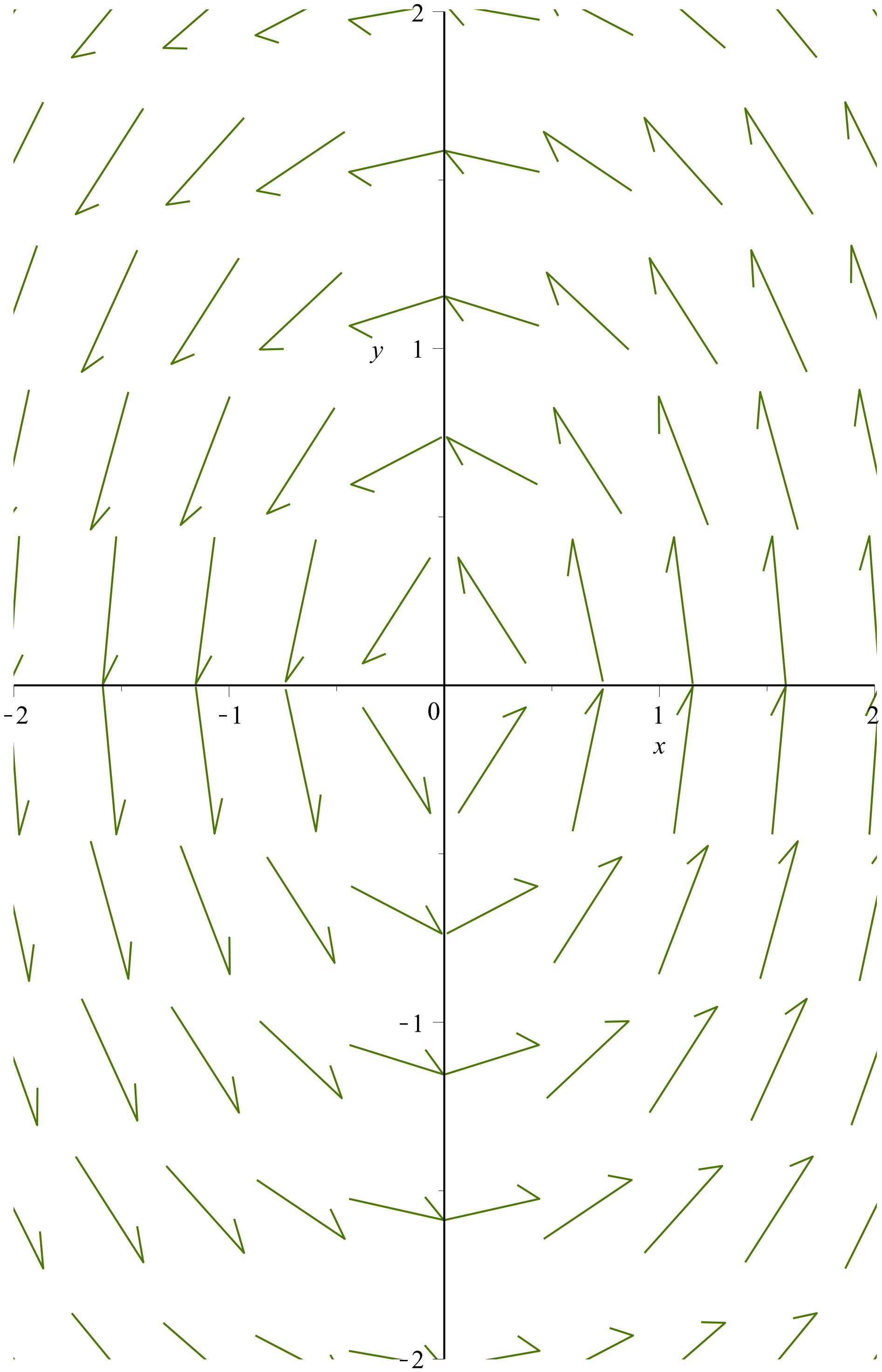
VectorField(⟨ −*y*, *x*⟩ , *cartesian*[*x*, *y*], output = plot, fieldoptions = [*grid* = [10, 10]], *axes* = *normal*);



2
1
y
-1
-2
-2
-1
0
1
2
x

S_2, |S_2| = 1

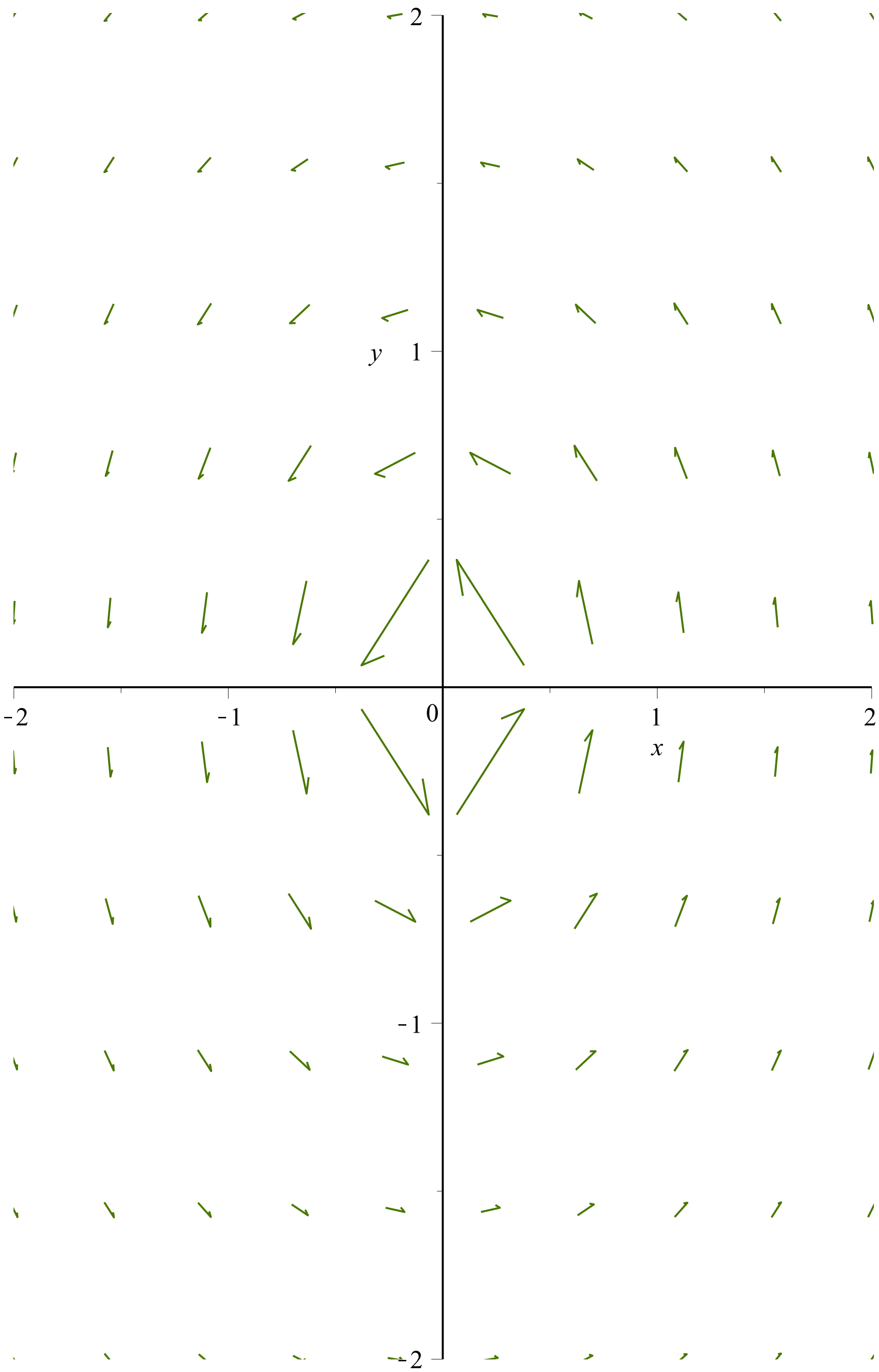
$$VectorField\left(\frac{\langle -y, x\rangle}{\operatorname{sqrt}(x^2+y^2)}, cartesian[x, y], \text{output} = \text{plot}, \text{fieldoptions} = [grid = [10, 10]], axes = normal\right);$$



2
y
1
-2
-1
0
1
2
x
-1
-2

S_3, |S_3| = 1/r

$VectorField\left(\frac{\langle -y, x\rangle}{x^2+y^2}, cartesian[x, y], \text{output} = \text{plot}, \text{fieldoptions} = [grid = [10, 10]], axes = normal\right);$

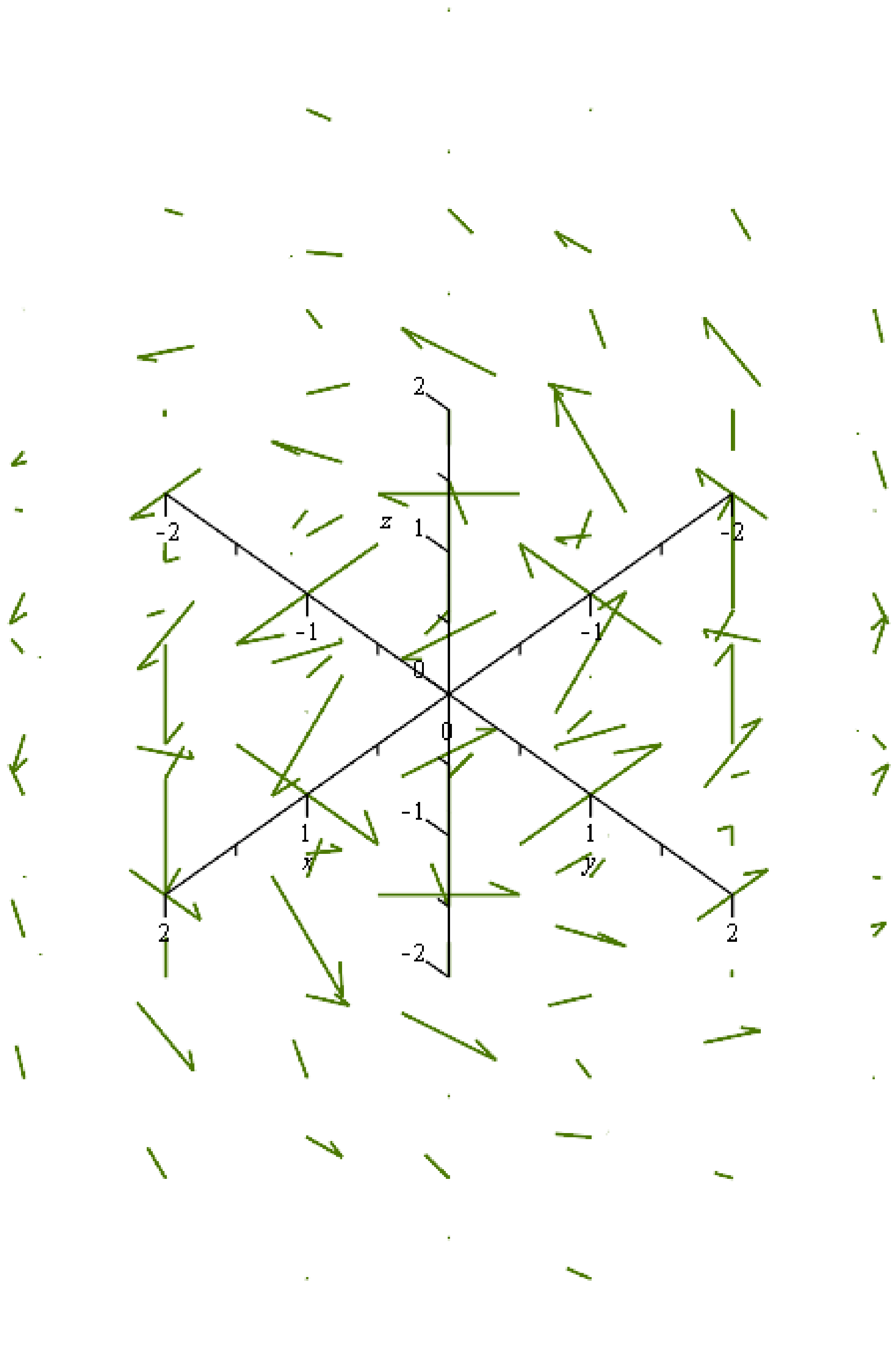


2
y
1
-1
-2
-2
-1
0
1
2
x

i 3D:

S3d_3, |S3d_3| = r

$VectorField\left(\frac{\langle -y, x, z\rangle}{x^2+y^2+z^2}, cartesian[x, y, z], \mathrm{output} = \mathrm{plot}, \mathrm{fieldoptions} = [grid = [5, 5, 5]], axes = normal \right);$



2
1
0
-1
-2
z
-2
-1
0
1
2
x
-2
-1
1
2
y

Gradientfelter (gradient fields)

f

$f := (x, y, z) \rightarrow 10 \cdot x \cdot y \cdot z - \frac{z-y}{x^2+1};$

$(x, y, z) \rightarrow 10\, x\, y\, z + Student{:}\text{-}VectorCalculus{:}\text{-}`\text{-}`\left(\left(z + Student{:}\text{-}VectorCalculus{:}\text{-}`\text{-}`(y)\right) \frac{1}{x^2+1} \right)$ **(4.1)**

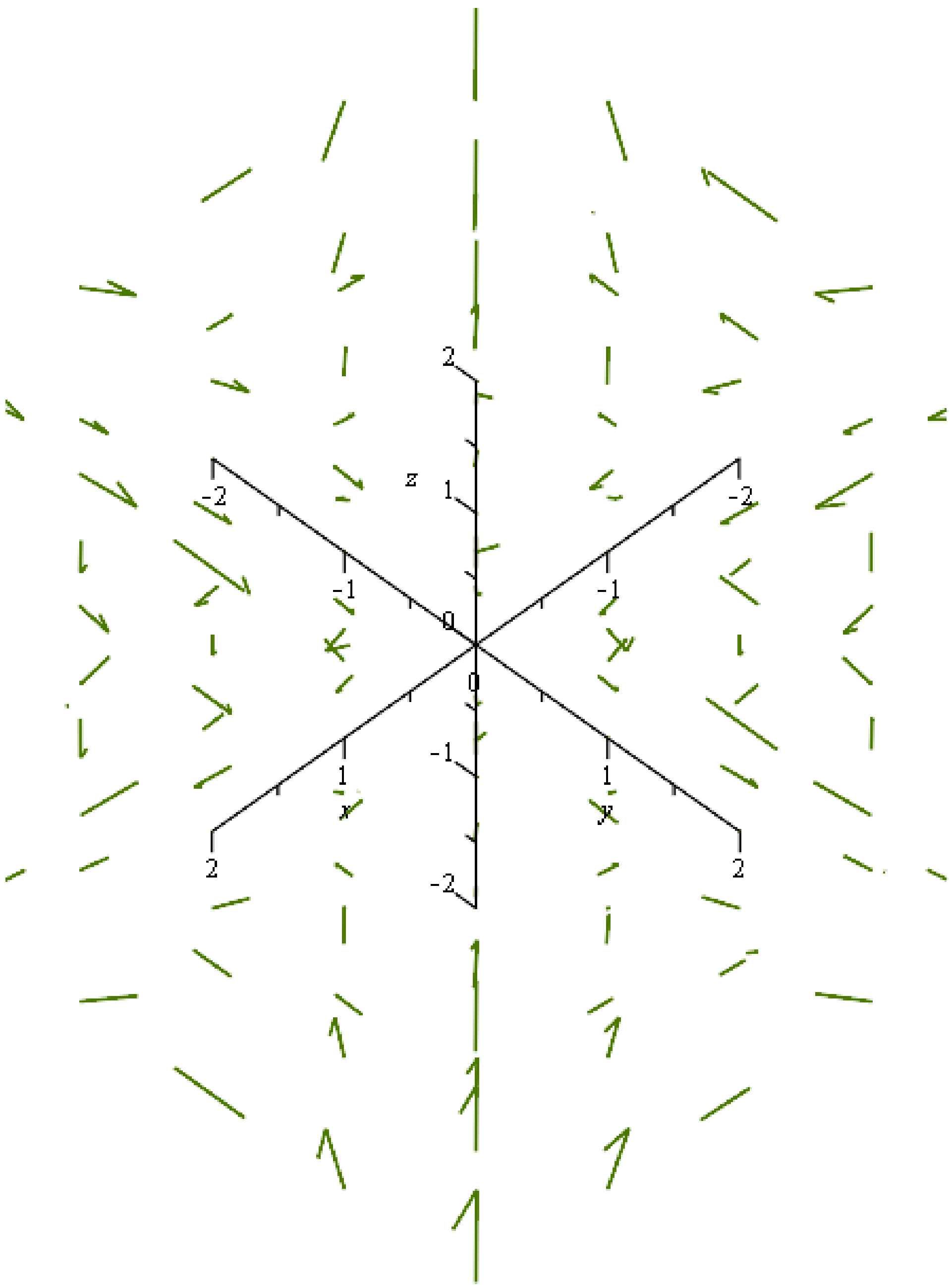
Gradienten til f

$Gradient(f(x, y, z), [x, y, z]);$

$$\begin{bmatrix} 10\, y\, z + \frac{2\,(z-y)\,x}{(x^2+1)^2} \\ 10\, x\, z + \frac{1}{x^2+1} \\ 10\, x\, y - \frac{1}{x^2+1} \end{bmatrix} \quad \textbf{(4.2)}$$

Gradientfeltet

$VectorField(\ Gradient(f(x, y, z), [x, y, z])\ , cartesian[x, y, z], output = 'plot', axes = normal);$



r_1

$r_1 := (x, y) \rightarrow \frac{1}{2} \cdot (x^2 + y^2);$

$$(x, y) \rightarrow 1 \frac{1}{2} x^2 + 1 \frac{1}{2} y^2 \quad \textbf{(4.3)}$$

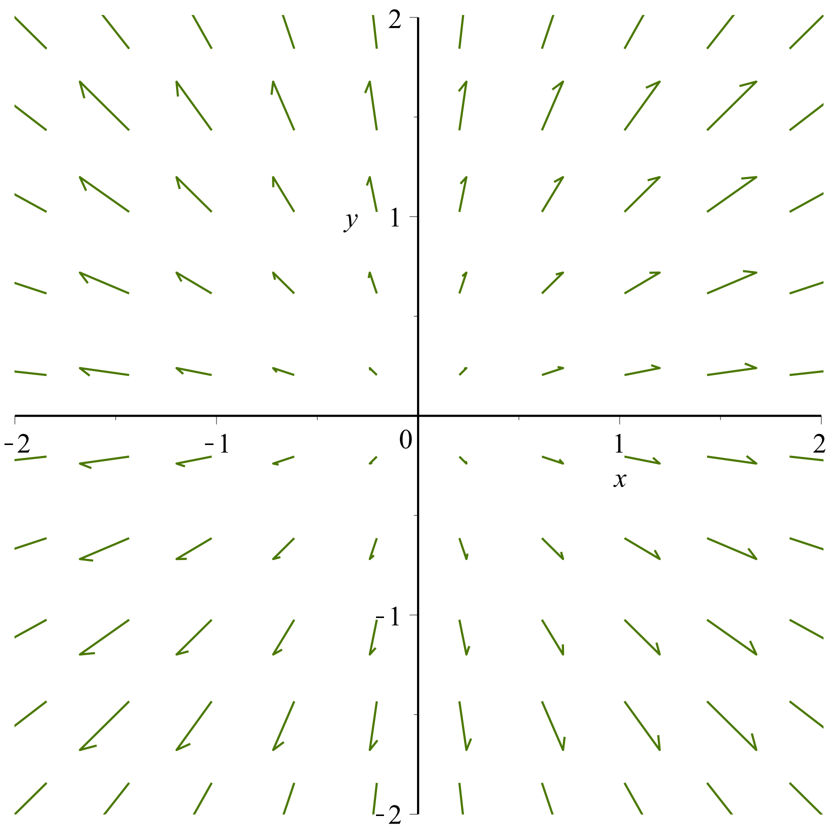
Gradienten til r_1

$Gradient(r_1(x, y), [x, y]);$

$$\begin{bmatrix} x \\ y \end{bmatrix} \quad \textbf{(4.4)}$$

Vi får R_1

$VectorField(\ Gradient(r_1(x, y), [x, y])\ , cartesian[x, y], \text{output} = \text{plot}, \text{fieldoptions} = [grid = [10, 10]], axes = normal);$



SPØRSMÅL: er alle radialfelter gradientfelter?
SVAR: JA

s_3

$s_3 := (x, y) \rightarrow \arctan\left(\frac{y}{x}\right);$

$$(x, y) \rightarrow \arctan\left(y \frac{1}{x}\right) \tag{4.5}$$

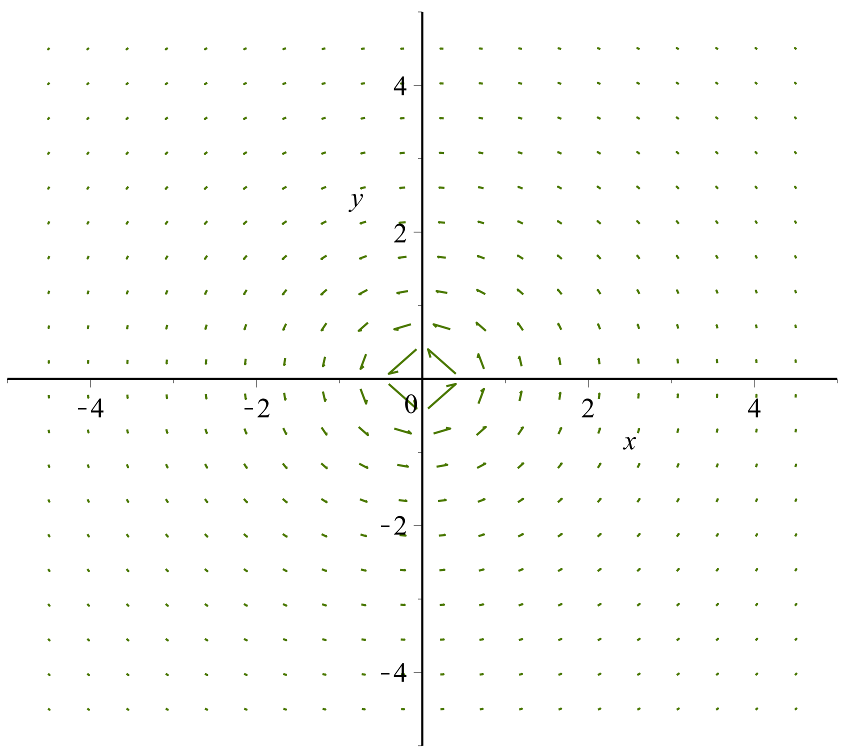
Gradienten til s_3

Gradient$(s_3(x, y), [x, y])$;

$$\begin{bmatrix} -\dfrac{y}{x^2\left(1+\dfrac{y^2}{x^2}\right)} \\ \dfrac{1}{x\left(1+\dfrac{y^2}{x^2}\right)} \end{bmatrix} \tag{4.6}$$

Vi får S_3

VectorField(*Gradient*$(s_3(x, y), [x, y])$, *cartesian*$[x, y]$, output = plot, fieldoptions = [*grid* = [10, 10]], *axes* = *normal*);



Arrows of the vector field, and the flow line(s) emanating from the given initial point(s)

SPØRSMÅL: er alle rotasjonsfelter gradientfelter?
SVAR: NEI, bare S_3

Eksempel 1

Finn arbeidet i kraftfeltet F=<y,3x> langs den øvre halvparten av ellipsen x^2/4 + y^2 = 1 fra (2,0) til (-2,0)

Kraftfeltet

Kraftfelt := *VectorField*(⟨*y*, 3 *x*⟩ , *cartesian*[*x*, *y*], *output* ='*plot*', fieldoptions = [*grid* = [10, 10]], *axes* = *normal*);

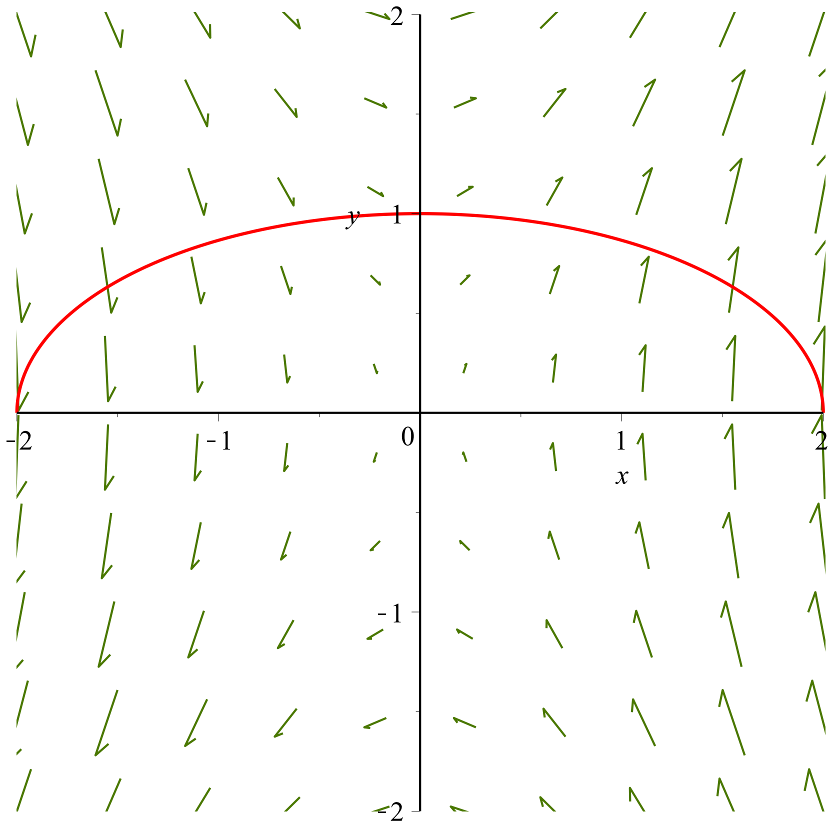
$$PLOT(\ldots) \tag{5.1}$$

Ellipsen

Ellipse := *plot*([2·cos(*t*), sin(*t*), *t* = 0 ..Pi], *color* = *red*);

$$PLOT(\ldots) \tag{5.2}$$

display(*Kraftfelt*, *Ellipse*);



Arbeidet langs kurven

LineInt(*VectorField*(⟨*y*, 3·*x*⟩), *Path*(⟨2·cos(*t*), sin(*t*)⟩ , *t* = 0 ..Pi), *output* = *integral*);

$$\int_0^{\pi} \left(-2 \sin(t)^2 + 6 \cos(t)^2\right) \mathrm{d}t \tag{5.3}$$

LineInt(*VectorField*(⟨*y*, 3·*x*⟩), *Path*(⟨2·cos(*t*), sin(*t*)⟩ , *t* = 0 ..Pi), *output* = *value*);

$$2\,\pi \tag{5.4}$$

Arbeidet langs x-aksen fra (2,0) til (-2,0)
$LineInt(VectorField(\langle y, 3\cdot x\rangle), Path(\langle -t, 0\rangle, t=2..-2), output=integral);$

$$\int_{2}^{-2} 0\, dt \tag{5.5}$$

Eksamensoppgave 2006 sommer / 5 - 1. gang (bare skriv opp)

Finn arbeidet i kraftfeltet F=<y,x,1> langs den orienterte kurven x=cos t - 1, y=cos^3 t + t, z = tan t
0 <= t <= Pi / 4

Kurven
$r := t \rightarrow \langle \cos(t) - 1, (\cos(t))^3 + t, \tan(t)\rangle;$

$$t \rightarrow Student{:-}VectorCalculus{:-}{<,>}(\cos(t) + (-1), \cos(t)^3 + t, \tan(t)) \tag{6.1}$$

Hastighetsvektoren
$diff(r(t), t);$

$$[-\sin(t), -3\cos(t)^2 \sin(t) + 1, 1 + \tan(t)^2] \tag{6.2}$$

Vektorfeltet
$Vektorfelt := (x, y, z) \rightarrow \langle y, x, 1\rangle;$

$$(x, y, z) \rightarrow Student{:-}VectorCalculus{:-}{<,>}(y, x, 1) \tag{6.3}$$

Vektorfeltet langs kurven
$Vektorfelt(\cos(t) - 1, \cos(t)^3 + t, \tan(t));$

$$(\cos(t)^3 + t)e_x + (\cos(t) - 1)e_y + e_z \tag{6.4}$$

F(r(t)) * dr/dt
$DotProduct(\langle(\cos(t)^3 + t), (\cos(t) - 1), 1\rangle, \langle -\sin(t), -3\cos(t)^2\sin(t) + 1, 1 + \tan(t)^2\rangle);$

$$-(\cos(t)^3 + t)\sin(t) + (\cos(t) - 1)(-3\cos(t)^2\sin(t) + 1) + 1 + \tan(t)^2 \tag{6.5}$$

Arbeidet (kjør kommandoen, trykk på 'All steps')
$IntTutor\left(-(\cos(t)^3 + t)\sin(t) + (\cos(t) - 1)(-3\cos(t)^2\sin(t) + 1) + 1 + \tan(t)^2, t=0..\frac{Pi}{4}\right);$

$$\int_{0}^{\frac{1}{4}\pi} \left(-(\cos(t)^3 + t)\sin(t) + (\cos(t) - 1)(-3\cos(t)^2\sin(t) + 1) + 1 + \tan(t)^2\right) dt = \frac{5}{4} - \frac{1}{4}\sqrt{2} - \frac{1}{4}\pi + \frac{1}{8}\sqrt{2}\,\pi \tag{6.6}$$

Eksempel 3: Sirkulasjonen langs hele ellipsen i Eksempel 1

Finn sirkulasjonen til vektorfeltet F=<y,3x> langs ellipsen x^2/4 + y^2 = 1 (retning (2,0) -> (-2, 0) -> (2,0))

Kraftfeltet

$Kraftfelt := VectorField(\ \langle y, 3\,x\rangle\ , cartesian[x, y], output='plot', \text{fieldoptions} = [grid = [10, 10]], axes = normal);$

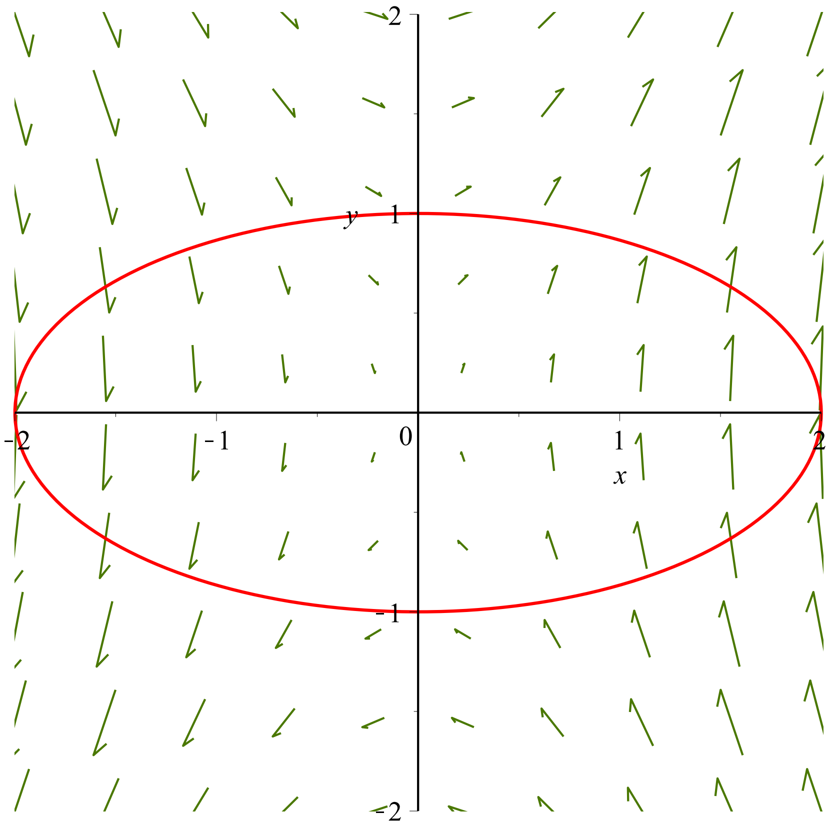
$$PLOT(\ldots) \qquad \textbf{(7.1)}$$

Ellipsen

$Ellipse := plot([2\cdot\cos(t), \sin(t), t = 0 ..2\cdot \text{Pi}], color = red);$

$$PLOT(\ldots) \qquad \textbf{(7.2)}$$

$display(Kraftfelt, Ellipse);$



Sirkulasjonen langs kurven

$LineInt(VectorField(\langle y, 3\cdot x\rangle), Path(\langle 2\cdot\cos(t), \sin(t)\rangle, t = 0 ..2\cdot \text{Pi}), output = integral);$

$$\int_0^{2\pi} \left(-2\sin(t)^2 + 6\cos(t)^2\right) \mathrm{d}t \qquad \textbf{(7.3)}$$

$LineInt(VectorField(\langle y, 3\cdot x\rangle), Path(\langle 2\cdot\cos(t), \sin(t)\rangle, t = 0 ..2\cdot \text{Pi}), output = value);$

$$4\,\pi \qquad \textbf{(7.4)}$$

Eksempel: Fluks gjennom en lukket kurve

Finn fluksen til F = R_3 = <x /r^2, y/ r^2> gjennom sirkelen C med radius a og sentrumet i origo

Vektorfeltet

$F := VectorField\left(\frac{\langle x, y\rangle}{x^2+y^2}, cartesian[x, y], output = 'plot', \text{fieldoptions} = [grid = [10, 10]], axes = normal \right);$

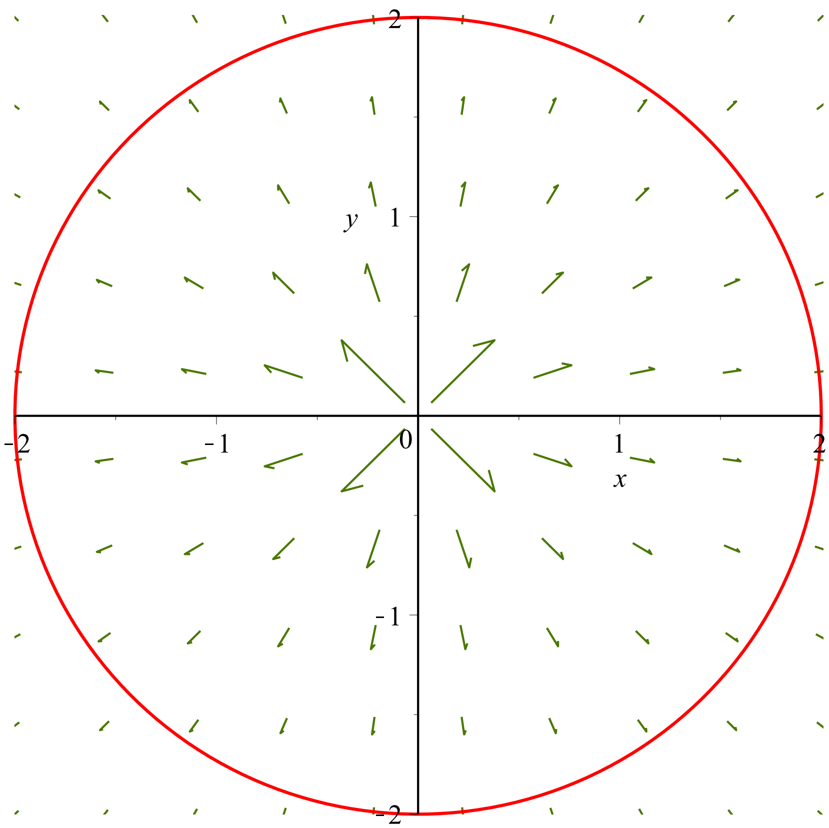
$$PLOT(\ldots) \qquad \textbf{(8.1)}$$

Sirkelen (a = 2) (t øker, vi går mot klokka)

$Sirkelen := plot([\, 2 \cdot \cos(t), 2 \cdot \sin(t), t = 0 ..2 \cdot \text{Pi}], color = red);$

$$PLOT(\ldots) \qquad \textbf{(8.2)}$$

$display(F, Sirkelen);$



Siden vi går mot klokka -> blir normalvektoren (som peker ut ac C) **T x k**

I)

$$Fluks := \int_C F \cdot n \, \mathrm{d}s = \int_C M \, \mathrm{d}y - N \, \mathrm{d}x$$

M = x /r^2
N = y / r^2
x = a * cos(t)
y = a * sin(t)

r = a
-->

M = a cos(t) / a^2
N = a sin(t) / a^2

dx = d(a cos(t)) = - a sin(t) dt
dy = d(a sin(t)) = a cos(t) dt

Derfor

$$Fluks := Int\left(\frac{a \cdot \cos(t)}{a^2} \cdot (a \cdot \cos(t)) - \frac{a \cdot \sin(t)}{a^2} \cdot (-a \cdot \sin(t)), t = 0 ..2 \cdot \mathrm{Pi}\right) = int\left(\frac{a \cdot \cos(t)}{a^2} \cdot (a \cdot \cos(t)) - \frac{a \cdot \sin(t)}{a^2} \cdot (-a \cdot \sin(t)), t = 0 ..2 \cdot \mathrm{Pi}\right);$$

$$\int_0^{2\pi} \left(\cos(t)^2 + \sin(t)^2\right) \mathrm{d}t = 2\pi \qquad \textbf{(8.3)}$$

II)

F * n = |F| * |n| * cosθ = |F| * 1 * cosθ

Vi kunne se i forkant at radialfeltet F er parallelt med normalvektoren, og vinkelen mellom F og n er 0, siden F peker ut av sirkelen
Derfor er F * n = |F| * 1 * 1 = |F| = 1/a   langs sirkelen

$$Fluks := \int_C F \cdot n \, \mathrm{d}s = \int_C |F| \mathrm{d}s = \int_C \frac{1}{a} \, \mathrm{d}s = \frac{1}{a} \int_C 1 \, \mathrm{d}s = \frac{1}{a} \cdot 2 \, \pi a = 2\pi$$