TMA4110 Calculus 3
Fall 2012

Norwegian University of Science and Technology
Department of Mathematical Sciences

Solutions to exercise set 10

$\boxed{1}$ The coordinate vector $[\mathbf{p}]_{\mathcal{B}}$ is a vector

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

such that $a(1+t^2)+b(t+t^2)+c(1+2t+t^2) = 3t-4$. This equation rewrites to $(a+c)+(b+2c)t+(a+b+c)t^2 = 3t-4$, giving the following system of equations:

$$\begin{aligned} a+c &= -4 \\ b+2c &= 3 \\ a+b+c &= 0 \end{aligned}$$

This is solved by reducing the augmented matrix:

$$\begin{bmatrix} 1 & 0 & 1 & -4 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{7}{2} \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix}$$

The coordinate vector is $[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} -\frac{7}{2} \\ 4 \\ -\frac{1}{2} \end{bmatrix}$.

$\boxed{2}$ We are given a matrix A, and the task is to find three bases – for the column space $\mathrm{Col}(A)$, the row space Row(A) and the null space $\mathrm{Nul}(A)$. Luckily, all of them can be found by reducing A to echelon form:

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -6 & 5 \\ 0 & 0 & 4 \\ -1 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \tilde{A}$$

A basis for the column space is the pivot columns of A, that is, the first and the last column (Theorem 6, page 212):

$$\mathcal{B}_{\mathrm{Col}(A)} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 4 \\ -3 \end{bmatrix} \right\}$$

A basis for the row space is simply the nonzero rows of $\tilde{A}$ (Theorem 13, page 231):

$$\mathcal{B}_{\mathrm{Row}(A)} = \left\{ \begin{bmatrix} 1 & -3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \right\}$$

To find a basis for the null space, we use the solution of the equation $A\mathbf{x} = \mathbf{0}$ (see page 211). From $\tilde{A}$, we see that x_2 is free, $x_1 = 3x_2$ and $x_3 = 0$. A general solution is

$$\mathbf{x} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

so the basis of the null space is

$$\mathcal{B}_{\mathrm{Nul}(A)} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$\boxed{3}$ The Rank Theorem (page 233) says that for any $m \times n$-matrix B, we have that $\mathrm{rank}(B) + \dim \mathrm{Nul}(B) = n$. In our situation, we have a 10×5-matrix A, so

$$\mathrm{rank}(A) + \dim \mathrm{Nul}(A) = 5$$

Moreover, we know a nonzero vector in the null space of A. Hence,

$$\dim \mathrm{Nul}(A) > 0$$

and the rank of A is at most 4.

$\boxed{4}$ We use Markov Chains, as described in section 4.9. The first thing we do, is to find the stochastic matrix P for the model.

$$\begin{array}{lcc} & \multicolumn{2}{c}{\textit{today}} \\ \textit{tomorrow} & \text{rain} & \text{not rain} \\ \text{rain} & 0.8 & 0.5 \\ \text{not rain} & 0.2 & 0.5 \end{array} = P$$

To find the probability that it will rain on a given day, we use the steady state-vector $\mathbf{q}$. This is a probability vector such that $P\mathbf{q} = \mathbf{q}$. We compute $\mathbf{q}$ by solving $(P - I)\mathbf{x} = \mathbf{0}$ (we know that $\mathbf{q}$ is a vector in the null space of $P - I$):

$$P - I = \begin{bmatrix} -0.2 & 0.5 \\ 0.2 & -0.5 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{5}{2} \\ 0 & 0 \end{bmatrix}$$

x_2 is free, so a solution is of the form $\mathbf{x} = \begin{bmatrix} \frac{5}{2} \\ 1 \end{bmatrix} x_2$. The solution we are looking for, $\mathbf{q}$, is a probability vector – that is, the sum of the entries is 1. We choose $x_2 = \frac{1}{\frac{5}{2}+1} = \frac{2}{7}$. This gives

$$\mathbf{q} = \begin{bmatrix} \frac{5}{7} \\ \frac{2}{7} \end{bmatrix}$$

and the probability that it will rain on a given day is $\frac{5}{7} \approx 0.71$.