TMA4110 Calculus 3
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Solutions to exercise set 11

$\boxed{1}$ We are given the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

To find the eigenvalues, we solve the characteristic equation:

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det \begin{bmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} &= 0 \\ \lambda^2(3-\lambda) - (3-\lambda) &= 0 \\ (3-\lambda)(\lambda^2 - 1) &= 0 \end{aligned}$$

So the eigenvalues are $3, 1$ and -1. The eigenspace corresponding to an eigenvalue λ is the null space of $A - \lambda I$. We find bases for the three eigenspaces of A:

1. $\lambda = 3$. We get

$$A - 3I = \begin{bmatrix} -3 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A general solution of $(A - 3I)\mathbf{x} = \mathbf{0}$ is (in coordinate notation) (x_1, x_2, x_3), where $x_1 = x_2 = 0$ and x_3 is free. Hence, $\{(0, 0, 1)\}$ is a basis for the eigenspace corresponding to $\lambda = 3$.

2. $\lambda = 1$. We get

$$A - I = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

A general solution of $(A - I)\mathbf{x} = \mathbf{0}$ is (x_1, x_2, x_3), where $x_3 = 0$, x_2 is free and $x_1 = x_2$. Hence, $\{(1, 1, 0)\}$ is a basis for the eigenspace corresponding to $\lambda = 1$.

3. $\lambda = -1$. We get

$$A + I = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

A general solution of $(A + I)\mathbf{x} = \mathbf{0}$ is (in coordinate notation) (x_1, x_2, x_3), where $x_3 = 0$, x_2 is free and $x_1 = -x_2$. Hence, $\{(-1, 1, 0)\}$ is a basis for the eigenspace corresponding to $\lambda = -1$.

$\boxed{2}$ We are given $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$, and the task is diagonalize A. We start by determining the eigenvalues and the eigenspaces, using the same procedure as in Problem 1.

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{bmatrix} = (\lambda - 5)(\lambda + 2)$$

The eigenspace corresponding to $\lambda = 5$ has basis $\left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$, and the eigenspace corresponding to $\lambda = -2$ has basis $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$. We form

$$P = \begin{bmatrix} 3 & -1 \\ 4 & 1 \end{bmatrix}$$

To find P^{-1} is easy, because P is a 2×2-matrix. We get

$$P^{-1} = \begin{bmatrix} 1/7 & 1/7 \\ -4/7 & 3/7 \end{bmatrix}$$

According to Theorem 5, page 282, A is equal to PDP^{-1} where $D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$.

$\boxed{3}$ We have the formula $x_i = Px_{i-1}$. That is, $x_1 = Px_0$, $x_2 = Px_1 = P(Px_0) = P^2x_0$, $x_3 = Px_2 = P(P^2x_0) = P^3x_0$, so we have that

$$x_i = P^i x_o$$

If P has two distinct eigenvalues λ_1 and λ_2, the corresponding eigenvectors $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent and hence span $\mathbb{R}^2$. Then we can find scalars a and b such that $\mathbf{x_0} = a\mathbf{v_1} + b\mathbf{v_2}$, and we have that

$$\begin{aligned}
\mathbf{x_1} &= P\mathbf{x_0} = P(a\mathbf{v_1} + b\mathbf{v_2}) = aP\mathbf{v_1} + bP\mathbf{v_2} = a\lambda_1\mathbf{v_1} + b\lambda_2\mathbf{v_2} \\
\mathbf{x_2} &= P\mathbf{x_1} = P(a\lambda_1\mathbf{v_1} + b\lambda_2\mathbf{v_2}) = a\lambda_1P\mathbf{v_1} + b\lambda_2P\mathbf{v_2} = a\lambda_1^2\mathbf{v_1} + b\lambda_2^2\mathbf{v_2} \\
&\vdots \\
\mathbf{x_i} &= a\lambda_1^i\mathbf{v_1} + b\lambda_2^i\mathbf{v_2}
\end{aligned}$$

(See example 5, page 278). We find the eigenvalues and eigenvectors of P:

$$\det(P - \lambda I) = (\lambda - 1)(\lambda - 0.3)$$

So $\lambda_1 = 1$ and $\lambda_2 = 0.3$. By the procedure from Problem 1, we find the corresponding eigenvectors:

$$\mathbf{v_1} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \text{ and } \mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Next, we have to compute a and b. This is done by reducing

$$\begin{bmatrix} 3 & -1 & 0.5 \\ 4 & 1 & 0.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{7} \\ 0 & 1 & -\frac{1}{14} \end{bmatrix}$$

The formula for $\mathbf{x_i}$ is

$$\mathbf{x_i} = a\lambda_1^i\mathbf{v_1} + b\lambda_2^i\mathbf{v_2} = \frac{1}{7} \cdot 1 \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} - \frac{1}{14} \cdot 0.3^i \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{7} + 0.3^i\frac{1}{14} \\ \frac{4}{7} - 0.3^i\frac{1}{14} \end{bmatrix}$$