Norwegian University of Science and Technology
Department of Mathematical Sciences

Solutions to exercise set 12

1 We have a system of differential equations on the form $\mathbf{x}'(t) = A\mathbf{x}(t)$, where

$$A = \begin{bmatrix} 17 & -15 \\ 20 & -18 \end{bmatrix}$$

We will use the method given on page 315 in the book. We perform the substitution $\mathbf{x}(t) = P\mathbf{y}(t)$, where P is a matrix which diagonalizes A. To find P, we need to find the eigenvalues and corresponding eigenvectors.

$$\det(A - \lambda I) = \det \begin{bmatrix} 17-\lambda & -15 \\ 20 & -18-\lambda \end{bmatrix} = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$$

The eigenvalue $\lambda = -3$ corresponds to the eigenvector $\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$, and the eigenvalue $\lambda = 2$ corresponds to the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. (This is easy; the procedure is described in detail in the soluions to the previous exercise set.) Now we have the matrix

$$P = \begin{bmatrix} 3/4 & 1 \\ 1 & 1 \end{bmatrix}$$

and we can do the substitution and simplify the new equation:

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} \\ (P\mathbf{y})' &= A(P\mathbf{y}) \\ P\mathbf{y}' &= AP\mathbf{y} \\ P^{-1}P\mathbf{y}' &= P^{-1}AP\mathbf{y} \\ \mathbf{y}' &= D\mathbf{y} \end{aligned}$$

where D is the diagonal matrix $\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$. Because D is of this form, the general solution to $\mathbf{y}'(t) = D\mathbf{y}(t)$ is

$$\begin{aligned} y_1(t) &= c_1 e^{-3t} \\ y_2(t) &= c_2 e^{2t} \end{aligned}$$

To find $\mathbf{x}$, recall that $\mathbf{x} = P\mathbf{y}$. This gives

$$\mathbf{x} = \begin{bmatrix} 3/4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-3t} \\ c_2 e^{2t} \end{bmatrix} = \begin{bmatrix} \frac{3}{4}c_1 e^{-3t} + c_2 e^{2t} \\ c_1 e^{-3t} + c_2 e^{2t} \end{bmatrix}$$

Finally, we need to find the constants c_1 and c_2. We have that $x_1(0) = 4$ and that $x_2(0) = 5$. Inserting this, we get the equations

$$\frac{3}{4}c_1 + c_2 = 4$$
$$c_1 + c_2 = 5$$

It is easy to find the soluions $c_1 = 4$ and $c_2 = 1$. Hence, the desired solution is

$$x_1(t) = 3e^{-3t} + e^{2t}$$
$$x_2(t) = 4e^{-3t} + e^{2t}$$

$\boxed{2}$ In this exercise, we are looking for the distance between the vectors $\mathbf{y}$ and $\hat{\mathbf{y}}$, where $\hat{\mathbf{y}}$ is the orthogonal projection of $\mathbf{y}$ onto the line through $\mathbf{u}$ and the origin. To find $\hat{\mathbf{y}}$, we use the formula on page 340 in the book:

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{15}{5}\mathbf{u} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

The distance is

$$||\mathbf{y} - \hat{\mathbf{y}}|| = \sqrt{(-6)^2 + 3^2} = \sqrt{45} \approx 6.7$$

$\boxed{3}$ We are given three vectors, and the first task is to normalize them to make an orthonormal basis form $\mathbb{R}^3$ (note that the vectors are orthogonal and linearly independent, this is easy to see). A normal vector is a vector with length (norm) equal to 1. Hence, to normalize a vector, we divide each entry with the norm. This gives the orthonormal basis

$$\mathbf{u_1} = \begin{bmatrix} 1/(3\sqrt{2}) \\ 4/(3\sqrt{2}) \\ 1/(3\sqrt{2}) \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \mathbf{u_3} = \begin{bmatrix} -2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

Next, we are given a vector $\mathbf{x}$ of $\mathbb{R}^3$, and we want to find the coordinate vector of $\mathbf{x}$ with respect to $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$. Theorem 5, page 339, says that

$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{u_1})\mathbf{u_1} + (\mathbf{x} \cdot \mathbf{u_2})\mathbf{u_2} + (\mathbf{x} \cdot \mathbf{u_3})\mathbf{u_3}$$

(the formula simplifies because the basis is orthonormal). Thus, we get

$$\mathbf{x} = \frac{7\sqrt{2}}{3}\mathbf{u_1} + \sqrt{2}\mathbf{u_2} - \frac{1}{3}\mathbf{u_3}$$