TMA4110 Calculus 3
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Solutions to exercise set 13

1 We have a matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \\ 1 & -2 \end{bmatrix}$ and a vector $\mathbf{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$.

Recall (Theorem 3, page 335) that the orthogonal complement of the column space of A is $\text{Nul}(A^T)$. Then Theorem 8, page 348, tells us that $\mathbf{y}$ can be uniquely written as $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, where $\hat{\mathbf{y}}$ is in $\text{Col}(A)$ and $\mathbf{z}$ is in $\text{Nul}(A^T)$. Moreover, the theorem tells us how to find $\hat{\mathbf{y}}$, given that we have an orthogonal basis for $\text{Col}(A)$. It is easily observed that the columns of A are orthogonal, so they form such a basis. To follow the notation of the theorem, we put

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{u_2} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$$

and then we compute

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u_1}}{\mathbf{u_1} \cdot \mathbf{u_1}} \mathbf{u_1} + \frac{\mathbf{y} \cdot \mathbf{u_2}}{\mathbf{u_2} \cdot \mathbf{u_2}} \mathbf{u_2} = \frac{6}{3} \mathbf{u_1} + \frac{7}{14} \mathbf{u_2} = \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix}$$

Since $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, we get

$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} -5/2 \\ 1/2 \\ 2 \end{bmatrix}$$

2 Given a matrix A with linearly independent columns, we can find the QR-factorization of A by the following procedure (following Example 4, page 357-358):

First, we need an orthonormal basis for $\text{Col}(A)$, which we will use to form the matrix Q. We observe that the columns $\mathbf{x_1}$ and $\mathbf{x_2}$ of A are not orthogonal, so we have to use the Gram-Schmidt process to find an orthogonal basis for $\text{Col}(A)$. This procedure is given by Theorem 11, page 355, and works as follows:

First we put $\mathbf{v_1} = \mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Then we compute

$$\mathbf{v_2} = \mathbf{x_2} - \frac{\mathbf{x_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}} \mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Now we have the orthogonal basis $\{\mathbf{v_1}, \mathbf{v_2}\}$ for $\mathrm{Col}(A)$. But we need an ortho*normal* basis, so we normalize $\mathbf{v_1}$ and $\mathbf{v_2}$:

$$\mathbf{u_1} = \frac{1}{\|\mathbf{v_1}\|}\mathbf{v_1} = \frac{1}{\sqrt{3}}\mathbf{v_1} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\mathbf{u_2} = \frac{1}{\|\mathbf{v_2}\|}\mathbf{v_2} = \frac{1}{\sqrt{2}}\mathbf{v_2} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

We form Q from the orthonormal basis:

$$Q = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix}$$

Since Q has orthonormal columns, we know that $Q^TQ = I$, and moreover since $A = QR$, we have that

$$\begin{aligned} QR &= A \\ Q^TQR &= Q^TA \\ R = Q^TA &= \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{2} \end{bmatrix} \end{aligned}$$

3 We have a set of four data points in $\mathbb{R}^2$, and we want to find the least-squares line $y = \beta_0 + \beta_1 x$ that best fits the data. We solve the problem by solving the normal equations, but to find the normal equations, we need to express the problem in terms of a matrix equation (see pages 368–370). First, we form the design matrix

$$X = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \\ 1 & 6 \end{bmatrix}$$

and the observation vector

$$\mathbf{y} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Now, we can express the problem as: Find the least-squares solution of $X\boldsymbol{\beta} = \mathbf{y}$. The normal equations are $X^TX\boldsymbol{\beta} = X^T\mathbf{y}$. We compute

$$X^TX = \begin{bmatrix} 4 & 16 \\ 16 & 74 \end{bmatrix}$$

and

$$X^T\mathbf{y} = \begin{bmatrix} 6 \\ 17 \end{bmatrix}$$

so the system we need to solve is

$$\begin{bmatrix} 4 & 16 \\ 16 & 74 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 6 \\ 17 \end{bmatrix}$$

By your favourite method of solving this system, the solution turns out to be

$$\beta_0 = \frac{43}{10} \quad \text{and} \quad \beta_1 = -\frac{7}{10}$$

so the best-fitting line is $y = \frac{43}{10} - \frac{7}{10}x$.