TMA4110 Calculus 3
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Solutions to exercise set 2

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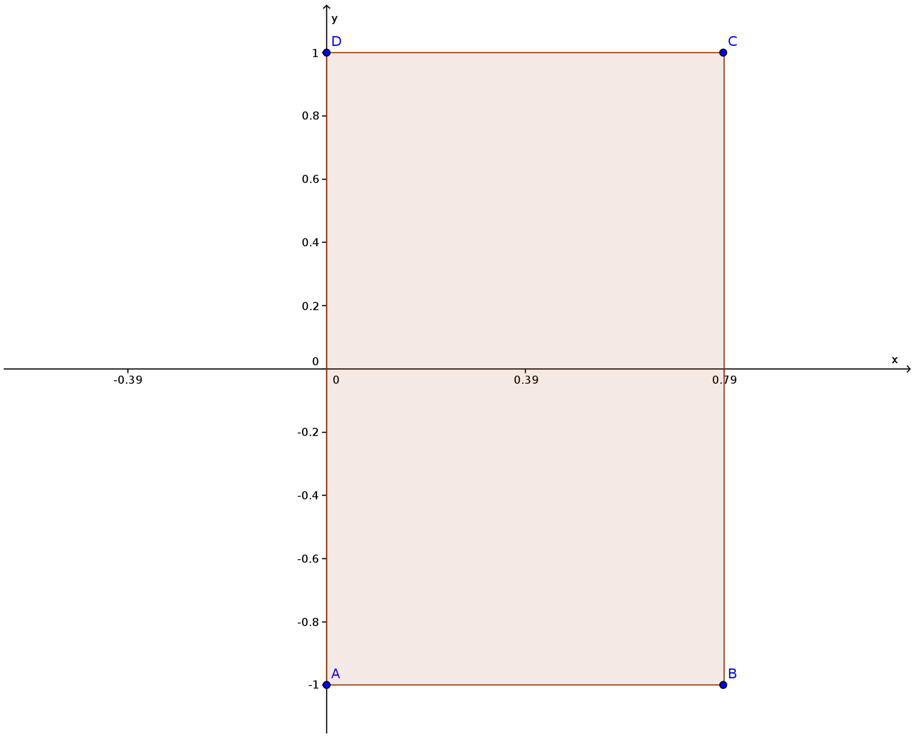
$\boxed{1}$ We have a set $\mathcal{D}$ of complex numbers, described as all complex numbers z such that $0 \leq \text{Re}(z) \leq \frac{\pi}{4}$ and $-1 \leq \text{Im}(z) \leq 1$. A picture of $\mathcal{D}$ will look like this:



We have (p. xxiii in the book) that $e^w = e^{\text{Re}(w)}(\cos \text{Im}(w) + i \sin \text{Im}(w))$. Hence, transforming $z = \text{Re}(z) + i\,\text{Im}(z)$ by f gives

$$f(z) = e^{2iz} = e^{-2\,\text{Im}(z)+2i\,\text{Re}(z)} = e^{-2\,\text{Im}(z)}(\cos(2\,\text{Re}(z)) + i\sin(2\,\text{Re}(z)))$$

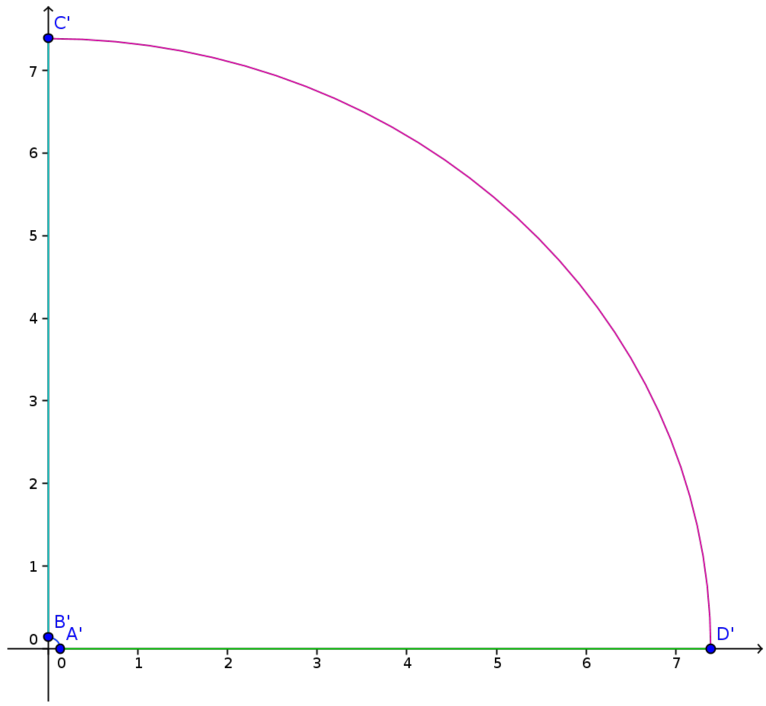
Hence the modulus of $f(z)$ is

$$|f(z)| = e^{-2\,\text{Im}(z)}$$

and the argument of $f(z)$ is

$$\arg(f(z)) = 2\,\text{Re}(z)$$

So $\mathcal{R}$ is the set of all complex numbers z such that $e^{-2} \leq |z| \leq e^2$ and $0 \leq \arg \leq \pi/2$. A picture of this set will look like this (where A' is $f(A)$ and so on):



2 We are given a polynomial $P(z) = z^4 - 2z^3 + 3z^2 - 2z + 2$ and a complex number $z_1 = i$. We check that z_1 is a root of P:

$$P(z_1) = i^4 - 2i^3 + 3i^2 - 2i + 2 = 1 + 2i - 3 - 2i + 2 = 0$$

So z_1 is a root. Then we proceed to find the other roots. Since $P(z)$ has real coefficients, we know that $\overline{z_1} = -i$ is also a root. Then $(z-i)(z+i) = z^2 + 1$ is a factor of $P(z)$. Polynomial division gives that

$$P(z) = (z^2 + 1)(z^2 - 2z + 2)$$

To find the last two roots, we use the quadratic formula:

$$\frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} = \frac{2 \pm 2i}{2}$$

which gives the two roots $1 + i$ and $1 - i$.

3 1. and 2. are linear, but only 2. is homogeneous. 3. is not linear because a function of y appears as a coefficient.

4 We are first supposed to show that $y_1 = e^t$ and $y_2 = te^t$ form a fundamental set of solutions for the differential equation $y'' - 2y' + y = 0$. We start by showing that both are solutions. For the first function, we have $y_1'' = y_1' = y_1$, which obviously gives zero when inserted into the equation. For the second function, we have $y_2' = e^t(1+t)$ and $y_2'' = e^t(2+t)$, which gives

$$e^t(2+t) - 2(e^t(1+t)) + te^t = 0$$

To show that they are linearly independent, we use the Wronski determinant:

$$W(t) = \begin{vmatrix} e^t & te^t \\ e^t & e^t(1+t) \end{vmatrix} = e^{2t}(1+t) - te^{2t} = e^{2t}$$

This is nonzero, so the functions are linearly independent. According to Theorem 1.23 (p. xli), y_1 and y_2 then form a fundamental set of solutions. Hence, a general solution for the differential equation is

$$y(t) = c_1e^t + c_2te^t$$

Next, we are supposed to find a solution y satisfying $y(0) = 1$ and $y'(0) = 0$. We compute $y'(t)$:

$$y'(t) = c_1e^t + c_2e^t(1+t)$$

Inserting the initial values, we get two equations

$$\begin{aligned} c_1 &= 2 \\ c_1 + c_2 &= 0 \end{aligned}$$

which gives $c_1 = 2$ and $c_2 = -2$. So the wanted solution is

$$y(t) = 2e^t - 2te^t$$