Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4110 Calculus 3
Fall 2012

Solutions to exercise set 4

$\boxed{1}$ We use elementary row operations to compute the reduced echelon form. The operations are listed. The pivot positions in the final matrix are circled, as well as the pivot posiotions of the original matrix. The pivot columns are columns 1, 2 and 3.

$$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 5 & 4 \\ 4 & 5 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 0 & -3 & -6 \\ 4 & 5 & 4 & 2 \end{bmatrix} \text{row2 } -2\cdot \text{ row1}$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 0 & -3 & -6 \\ 0 & -3 & -12 & -18 \end{bmatrix} \text{row3 } -4\cdot\text{row1}$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & -3 & -12 & -18 \\ 0 & 0 & -3 & -6 \end{bmatrix} \text{swap row2 and row3}$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & -3 & -6 \end{bmatrix} \text{scale row2 with } -\frac{1}{3}$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \text{scale row3 with } -\frac{1}{3}$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \text{row2 } -4\cdot \text{ row3}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \text{row1 } -4\cdot \text{ row3}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \text{row1 } -2\cdot \text{ row2}$$

$\boxed{2}$ We have a system with three unknowns and two equations. The augmented matrix of the system is

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ -2 & 4 & 2 & 6 \end{bmatrix}$$

By elementary row operations, the system is reduced to

$$\begin{bmatrix} 1 & 0 & -1 & 7 \\ 0 & 1 & 0 & 5 \end{bmatrix}$$

We see that $x_1 = 7 + x_3$ and $x_2 = 5$. We have no restrictions on x_3, so this is a free variable.

$\boxed{3}$ The given differential equation is $y'' + 4y' + 4y = t^{-2}e^{-2t}$. We use the technique of chapter 4.6.

The characteristic polynomial is $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2$, so $\lambda = -2$ is the only characteristic root. It follows that $y_h(t) = c_1 e^{-2t} + c_2 t e^{-2t}$ is the general solution to the homogeneous equation $y'' + 4y' + 4y = 0$.

Let $y_p(t) = v_1(t)e^{-2t} + v_2(t)te^{-2t}$. Then $y_p'(t) = v_1'(t)e^{-2t} - 2v_1(t)e^{-2t} + v_2'(t)te^{-2t} + v_2(t)e^{-2t} - 2v_2(t)te^{-2t}$. We assume that $v_1'(t)e^{-2t} + v_2'(t)te^{-2t} = 0$. Then $y_p'(t) = -2v_1(t)e^{-2t} + v_2(t)e^{-2t} - 2v_2(t)te^{-2t}$ and

$$\begin{aligned} y_p''(t) + 4y_p'(t) + 4y_p(t) &= -2v_1'(t)e^{-2t} + 4v_1(t)e^{-2t} + v_2'(t)e^{-2t} - 2v_2(t)e^{-2t} \\ &\quad - 2v_2'(t)te^{-2t} - 2v_2(t)e^{-2t} + 4v_2(t)te^{-2t} - 8v_1(t)e^{-2t} \\ &\quad + 4v_2(t)e^{-2t} - 8v_2(t)te^{-2t} + 4v_1(t)e^{-2t} + 4v_2(t)te^{-2t} \\ &= -2v_1'(t)e^{-2t} + v_2'(t)(1-2t)e^{-2t}. \end{aligned}$$

So y_p is a partial solution if $v_1'(t)e^{-2t} + v_2'(t)te^{-2t} = 0$ and $-2v_1'(t)e^{-2t} + v_2'(t)(1 - 2t)e^{-2t} = t^{-2}e^{-2t}$. Since $e^{-2t} \neq 0$ this is equivalent to $v_1'(t) + v_2'(t)t = 0$ and $-2v_1'(t) + v_2'(t)(1-2t) = t^{-2}$.

The solution to the system

$$\begin{aligned} v_1'(t) + v_2'(t)t &= 0 \\ -2v_1'(t) + v_2'(t)(1-2t) &= t^{-2} \end{aligned}$$

is $v_1'(t) = -t^{-1}$ and $v_2'(t) = t^{-2}$, so if

$$v_1(t) = -\int t{-}1 \, dt = -\ln(t)$$

and

$$v_2(t) = \int t^{-2} \, dt = -\frac{1}{t}$$

then $y_p(t) = v_1(t)e^{-2t} + v_2(t)te^{-2t} = -\ln(t)e^{-2t} - e^{-2t}$ is a partial solution. It follows that $y(t) = y_p(t) + y_h(t) = c_1 e^{-2t} + c_2 te^{-2t} - \ln(t)e^{-2t} - e^{-2t}$ is the general solution.

$\boxed{4}$ We have the equation

$$y'' + 2y' + 4y = 4\cos(2t) \tag{1}$$

The characteristic function is $P(\lambda) = \lambda^2 + 2\lambda + 4$, $\omega = 2$ and the transfer function is

$$H(i\omega) = H(2i) = \frac{1}{P(2i)} = \frac{1}{(2i)^2 + 2(2i) + 4} = \frac{1}{-4 + 4i + 4} = \frac{1}{4i} = \frac{-i}{4}$$

So $z(t) = H(i\omega)4e^{2it} = -ie^{2it}$ is a solution to the equation $z'' + 2z' + 4 = 4e^{2it}$. It follows that $y(t) = \mathrm{Re}(z(t)) = -\mathrm{Re}(ie^{2it}) = \sin(2t)$ is the steady-state solution to $y'' + 2y' + 4y = 4\cos(2t)$.